Math 220C - Lecture 23

May 27, 2022

Last time

$\mathcal{F} \to X$ flabby if $\forall\ V \subseteq U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \text{ surjective}$$

Key Lemma

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact, $\mathcal{F}$ flabby then

[I] $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact, $\forall\ U \subseteq X$ open

[II] $\mathcal{F}, \mathcal{G}$ flabby $\Rightarrow$ $\mathcal{H}$ flabby

We saw $\boxed{i} \Rightarrow \boxed{ii}$ last time.

We also saw we may take $U = X$ in $\boxed{i}$.

We showed $0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ exact in

Lecture 21.

Suffices to show

$$\beta : \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \text{ surjective if } \mathcal{F} \text{ flabby}$$

Proof Let $h \in \mathcal{H}(X)$. Define

$$\mathcal{A} = \{ (g, U) : g \in \mathcal{G}(U) \text{ and } \beta(g) = h/_U \}.$$

Define

$$(g, U) \geq (g', U') \text{ if } U \supseteq U' \text{ \& } g/_{U'} = g'$$

Remark Every linearly ordered chain admits an upper bound (take the union).

Zorn
$\Rightarrow$ $\mathcal{A}$ admits a maximal element $(g, \mathcal{U})$.

Claim $\mathcal{U} = X$ for the maximal $(g, \mathcal{U})$.

This gives $\beta : \mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ surjective.

Proof of the claim

(1) $\mathcal{U} \neq X$. We obtain a contradiction. Let $p \in X \setminus \mathcal{U}$.

(2) $\beta_p : \mathcal{G}_p \longrightarrow \mathcal{H}_p$ surjective $\Rightarrow$

$\Rightarrow \exists\ \tilde{g}_p$ with $\beta_p(\tilde{g}_p) = h_p$

$\Rightarrow \exists\ V\ \exists\ \tilde{g} \in \mathcal{G}(V),\ \beta(\tilde{g}) = h|_V$.

(3) Overlaps: $U \cap V$

$$\beta(\tilde{g}/_{U\cap V}) = \beta(g/_{U\cap V}) = h/_{U\cap V}$$

$$\Rightarrow \boxed{\tilde{g}/_{U\cap V} - g/_{U\cap V} = \alpha(f)}, \quad f \in \mathcal{F}(U\cap V).$$

This uses $0 \longrightarrow \mathcal{F}(U\cap V) \longrightarrow \mathcal{G}(U\cap V) \longrightarrow \mathcal{H}(U\cap V)$

exact, as proved in Lecture 23.

(4) $\mathcal{F}$ flabby $\Rightarrow$ extend f to X.

(5) Define $W = U \cup V$ and

$$\tilde{\tilde{g}} = \begin{cases} g & \text{in } \mathcal{U} \\ \tilde{g} - \alpha(f) & \text{in } V \end{cases} \qquad \text{well-defined}$$

Note $\beta(\tilde{\tilde{g}}) = h \Rightarrow (\tilde{\tilde{g}}, W) \in \mathcal{A}$.

This element contradicts maximality of $(g, \mathcal{U})$.

§2. The Godement Sheaf

Definition Given $\mathcal{F} \longrightarrow X$, define the sheaf $\Phi\mathcal{F}$ via

$$\Phi\mathcal{F}(U) = \prod_{x \in U} \mathcal{F}_x .$$

This is called the sheaf of totally discontinuous sections.

(the Godement sheaf).

Remarks

[1] We define $\mathcal{F} \longrightarrow \phi\mathcal{F}$ sending

$$f \longrightarrow (f_x)_{x \in U} \qquad \text{for } f \in \mathcal{F}(U).$$

We have $\mathcal{F} \longrightarrow \phi\mathcal{F}$ injective. Indeed,

$$f = 0 \iff f_x = 0 \quad \forall\, x \in U,$$ see Lecture 22

$\boxed{\text{ii}}$ $\Phi\mathcal{F}$ is flabby. Indeed, for $U \supseteq V$,

$$\phi\,\mathcal{F}(U) \longrightarrow \phi\,\mathcal{F}(V) \quad \text{surjective}$$

$$\Longleftrightarrow \quad \prod_{x \in U} \mathcal{F}_x \longrightarrow \prod_{x \in V} \mathcal{F}_x \quad \text{surjective, which is clear}$$

$\boxed{\text{iii}}$ If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact then

$$\Downarrow$$

$$0 \longrightarrow \phi\,\mathcal{F} \longrightarrow \phi\,\mathcal{G} \longrightarrow \phi\,\mathcal{H} \longrightarrow 0 \quad \text{exact}$$

Why? For all $U \subseteq X$ open, we have

$$0 \longrightarrow \phi\,\mathcal{F}(U) \longrightarrow \phi\,\mathcal{G}(U) \longrightarrow \phi\,\mathcal{H}(U) \longrightarrow 0 \quad \text{exact.}$$

Indeed, this is because

$$0 \longrightarrow \prod_{x \in U} \mathcal{F}_x \longrightarrow \prod_{x \in U} \mathcal{G}_x \longrightarrow \prod_{x \in U} \mathcal{H}_x \longrightarrow 0 \quad \text{exact}$$

which follows since $0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$ exact.

$\boxed{IV}$ Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$$\begin{tikzcd}
 & \text{exact} \arrow[d, bend left] & \text{exact} \arrow[d, bend left] & \text{exact} \arrow[d, bend left] & & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 & \swarrow \text{exact} \\
0 \arrow[r] & \phi\mathcal{F} \arrow[r] \arrow[d] & \phi\mathcal{G} \arrow[r] \arrow[d] & \phi\mathcal{H} \arrow[r] \arrow[d] & 0 & \swarrow \text{exact by } \boxed{III} \\
0 \arrow[r] & \phi\mathcal{F}/\mathcal{F} \arrow[r] \arrow[d] & \phi\mathcal{G}/\mathcal{G} \arrow[r] \arrow[d] & \phi\mathcal{H}/\mathcal{H} \arrow[r] \arrow[d] & 0 & \swarrow \Downarrow \text{ exact} \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

The last row is exact. This can be checked on stalks.

In the diagram of stalks, the columns & first 2 rows are exact $\Rightarrow$ 3^{rd} row is also exact.

The canonical flabby resolution

Theorem [i] Given $\mathcal{F} \to X$ we construct a resolution

$$(*) \qquad 0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

where $\mathcal{F}^p$ are flabby $\forall\, p \geq 0$.

[ii] If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact, then the Godement resolutions fit into exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dash] & \mathcal{F} \arrow[r, dash] \arrow[d] & \mathcal{G} \arrow[r, dash] \arrow[d] & \mathcal{H} \arrow[r, dash] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_0 \arrow[r] \arrow[d] & \mathcal{G}_0 \arrow[r, dash] \arrow[d] & \mathcal{H}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

Proof $\boxed{1}$ Form the exact sequences:

$$(0) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \phi\,\mathcal{F} \longrightarrow \tilde{\mathcal{F}}^{1} = \phi\,\mathcal{F}/\mathcal{F} \longrightarrow 0$$

$$(1) \qquad 0 \longrightarrow \tilde{\mathcal{F}}^{1} \longrightarrow \phi\,\tilde{\mathcal{F}}^{1} \longrightarrow \tilde{\mathcal{F}}^{2} = \phi\,\tilde{\mathcal{F}}^{1}/\tilde{\mathcal{F}}^{1} \longrightarrow 0$$

$$\vdots$$

$$(p) \qquad 0 \longrightarrow \tilde{\mathcal{F}}^{p} \longrightarrow \phi\,\tilde{\mathcal{F}}^{p} \longrightarrow \tilde{\mathcal{F}}^{p+1} = \phi\,\tilde{\mathcal{F}}^{p}/\tilde{\mathcal{F}}^{p} \longrightarrow 0$$

where we define

- $\tilde{\mathcal{F}}^{0} = \mathcal{F}$
- $\mathcal{F}^{p} = \phi\,\tilde{\mathcal{F}}^{p}$ = flabby
- $\tilde{\mathcal{F}}^{p+1} = \phi\,\tilde{\mathcal{F}}_{p}/\tilde{\mathcal{F}}_{p}$

The resolution (*) follows by concatenating the above exact sequences.

[II] Functoriality of the Godement resolution

Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

e.g. $0 \longrightarrow \widetilde{\mathcal{F}}^{0} \longrightarrow \widetilde{\mathcal{G}}^{0} \longrightarrow \widetilde{\mathcal{H}}^{0} \longrightarrow 0$ exact.

We show $0 \longrightarrow \widetilde{\mathcal{F}}^{p} \longrightarrow \widetilde{\mathcal{G}}^{p} \longrightarrow \widetilde{\mathcal{H}}^{p} \longrightarrow 0$ exact $\forall p$

We use induction on p. The case $p=0$ is clear.

For the inductive step, we use [IV] & diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \tilde{\mathcal{F}}^{p} \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^{p} \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^{p} \arrow[r] \arrow[d] & 0 & \text{exact} \\
0 \arrow[r] & \mathcal{F}^{p} \arrow[r] \arrow[d] & \mathcal{G}^{p} \arrow[r] \arrow[d] & \mathcal{H}^{p} \arrow[r] \arrow[d] & 0 & \text{exact by } \boxed{\mathrm{III}} \arrow[d, Rightarrow] \\
0 \arrow[r] & \tilde{\mathcal{F}}^{p+1} \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^{p+1} \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^{p+1} \arrow[r] \arrow[d] & 0 & \text{exact} \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

This also shows $0 \longrightarrow \mathcal{F}^{p} \longrightarrow \mathcal{G}^{p} \longrightarrow \mathcal{H}^{p} \longrightarrow 0$ exact.

Conclusion

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$$\Downarrow$$

$0 \longrightarrow \mathcal{F}^{\bullet} \longrightarrow \mathcal{G}^{\bullet} \longrightarrow \mathcal{H}^{\bullet} \longrightarrow 0$ exact.

$$\Downarrow \quad \text{Key Lemma}$$

$0 \longrightarrow \mathcal{F}^{\bullet}(X) \longrightarrow \mathcal{G}^{\bullet}(X) \longrightarrow \mathcal{H}^{\bullet}(X) \longrightarrow 0$ exact

§ 3. Main Theorem of Sheaf Cohomology

$\exists$ functors

$$H^p(X, -) : \text{Sheaves on } X \longrightarrow \text{Abelian Groups}$$

such that

$\boxed{a}$ $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

$\boxed{b}$ $\mathcal{F}$ flabby $\Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1$

$\boxed{c}$ Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

$\exists$ connecting homomorphisms

$$\delta_p : H^p(X, \mathcal{H}) \longrightarrow H^{p+1}(X, \mathcal{F})$$

functorial in exact sequences such that

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \xrightarrow{\delta_0} \\ \hookrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \xrightarrow{\delta_1} \\ \hookrightarrow \ldots \end{aligned}$$

exact.

These requirements determine the functors uniquely.

Aside from Homological Algebra

(1) Given a complex $d \circ d = 0$

$$\to A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \to A^n \to \cdots$$

we define

$$\boxed{\mathcal{H}^p(A^\bullet) = \frac{\mathrm{Ker}\; A^p \xrightarrow{d} A^{p+1}}{\mathrm{Im}\; A^{p-1} \xrightarrow{d} A^p}.}$$

(2) Given complexes $A^\bullet, B^\bullet, C^\bullet$ such that

$$0 \to A^p \to B^p \to C^p \to 0 \quad \text{exact} \quad \forall p$$

we write $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0.$ exact

(3) If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ exact then

$$\hookrightarrow \mathcal{H}^p(A^\bullet) \to \mathcal{H}^p(B^\bullet) \to \mathcal{H}^p(C^\bullet)$$

$$\hookrightarrow \mathcal{H}^{p+1}(A^\bullet) \to \mathcal{H}^{p+1}(B^\bullet) \to \mathcal{H}^{p+1}(C^\bullet)$$

$$\hookrightarrow \cdots$$

exact.

Outline of the argument

[i] every sheaf $\mathcal{F}$ admits a canonical flabby resolution

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \quad , \quad \mathcal{F}^p \text{ flabby}$$

[ii] Take global sections. We obtain the complex

$$\mathcal{F}^0(X) \to \mathcal{F}^1(X) \to \mathcal{F}^2(X) \to \cdots$$

[iii] Define

$$H^p(X, \mathcal{F}) = \frac{\operatorname{Ker} \ \mathcal{F}^p(X) \to \mathcal{F}^{p+1}(X)}{\operatorname{Im} \ \mathcal{F}^{p-1}(X) \to \mathcal{F}^p(X)}.$$

[iv] Show this works.