Math 220C - Lecture 24

June 1, 2022

Last time

[i] every sheaf $\mathcal{F}$ admits a canonical flabby resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \quad , \quad \mathcal{F}^p \text{ flabby}$$

[ii] Take global sections. We obtain the complex

$$\mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \mathcal{F}^2(X) \longrightarrow \cdots$$

[iii] Define

$$H^p(X, \mathcal{F}) = \frac{\operatorname{Ker} \ \mathcal{F}^p(X) \longrightarrow \mathcal{F}^{p+1}(X)}{\operatorname{Im} \ \mathcal{F}^{p-1}(X) \longrightarrow \mathcal{F}^p(X)} .$$

Main Theorem of Sheaf Cohomology

$\exists$ functors

$$H^p(X, -): \text{Sheaves on } X \longrightarrow \text{Abelian Groups}$$

such that

[a] $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

[b] $\mathcal{F}$ flabby $\Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1$

[c] Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

$\exists$ connecting homomorphisms

$$\delta_p : H^p(X, \mathcal{H}) \longrightarrow H^{p+1}(X, \mathcal{F})$$

functorial in exact sequences such that

$$\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{F}) &\longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \xrightarrow{\delta_0} \\
\hookrightarrow H^1(X, \mathcal{F}) &\longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \xrightarrow{\delta_1} \\
\hookrightarrow \quad \ldots &
\end{aligned}$$

exact.

Property $\boxed{a}$

<u>WTS</u> $\quad H^0(X, \mathcal{F}) = \operatorname{Ker} \mathcal{F}^0(X) \xrightarrow{\gamma\beta} \mathcal{F}^1(X) \overset{?}{=} \mathcal{F}(X).$

Break the resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots$$

into

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{F}^0 \xrightarrow{\beta} \mathcal{G}^0 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G}^0 \xrightarrow{\gamma} \mathcal{F}^1 \longrightarrow \mathcal{G}^1 \longrightarrow 0$$

Then

$$(0) \quad 0 \longrightarrow \mathcal{F}(X) \xrightarrow{\alpha} \mathcal{F}^0(X) \xrightarrow{\beta} \mathcal{G}^0(X)$$

$$(1) \quad 0 \longrightarrow \mathcal{G}^0(X) \xrightarrow{\gamma} \mathcal{F}^1(X) \longrightarrow \mathcal{G}^1(X)$$

$$\Rightarrow H^0(X, \mathcal{F}) = \operatorname{Ker} \gamma\beta \overset{\gamma \text{ injective}}{=} \operatorname{Ker} \beta \quad \text{by } (1)$$

$$= \operatorname{Im} \alpha \cong \mathcal{F}(X) \quad \text{by } (0).$$

Property [6]

WTS : $\mathcal{F}$ flabby $\implies H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1$.

Break the resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots$$

into

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{G}^0 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{G}^1 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \mathcal{G}^2 \longrightarrow 0$$

$$\cdots$$

$\mathcal{F}, \mathcal{F}^0$ flabby $\implies \mathcal{G}^0$ flabby by a lemma last time

By induction $\mathcal{G}^p$ is flabby $\forall p$. Indeed

$$0 \longrightarrow \mathcal{G}^p \longrightarrow \mathcal{F}^{p+1} \longrightarrow \mathcal{G}^{p+1} \longrightarrow 0 \quad \text{so}$$

$\mathcal{G}^p, \mathcal{F}^{p+1}$ flabby $\implies \mathcal{G}^{p+1}$ flabby.

Using $\mathcal{F}$ & $\mathcal{G}^p$ flabby, we also get exactness to the right:

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}^0(X) \longrightarrow \mathcal{G}^0(X) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \mathcal{G}^1(X) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G}^1(X) \longrightarrow \mathcal{F}^2(X) \longrightarrow \mathcal{G}^2(X) \longrightarrow 0$$

$$\cdots$$

Concatenating these exact sequences, we obtain

$$\mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \cdots \longrightarrow \mathcal{F}^p(X) \longrightarrow \cdots$$

exact $\Rightarrow$ $H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$

Aside from Homological Algebra

(1) Given a complex $d \circ d = 0$

$$\to A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \longrightarrow A^n \longrightarrow \cdots$$

we define $$\boxed{H^p(A^\bullet) = \frac{\mathrm{Ker}\ A^p \xrightarrow{d} A^{p+1}}{\mathrm{Im}\ A^{p-1} \xrightarrow{d} A^p}}.$$

(2) Given complexes $A^\bullet, B^\bullet, C^\bullet$ such that

$$0 \longrightarrow A^p \longrightarrow B^p \longrightarrow C^p \longrightarrow 0 \quad \text{exact} \quad \forall p$$

we write $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$. exact

(3) If $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ exact then

$$\hookrightarrow H^p(A^\bullet) \longrightarrow H^p(B^\bullet) \longrightarrow H^p(C^\bullet)$$

$$\hookrightarrow H^{p+1}(A^\bullet) \longrightarrow H^{p+1}(B^\bullet) \longrightarrow H^{p+1}(C^\bullet)$$

$$\hookrightarrow \cdots$$

exact.

Property c

Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$\Downarrow$ last time

$0 \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathcal{F}^\bullet(X) \longrightarrow \mathcal{G}^\bullet(X) \longrightarrow \mathcal{H}^\bullet(X) \longrightarrow 0$ exact

(here we use $\mathcal{F}^\bullet$ are flabby sheaves & Key lemma).

$$\Rightarrow \quad \hookrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{G}) \longrightarrow H^p(X, \mathcal{H})$$
$$\hookrightarrow H^{p+1}(X, \mathcal{F}) \longrightarrow \cdots$$

using the facts from homological algebra reviewed above for

$$A^\bullet = \mathcal{F}^\bullet(X), \quad B^\bullet = \mathcal{G}^\bullet(X), \quad C^\bullet = \mathcal{H}^\bullet(X).$$

Sheaves of modules

Let $\mathcal{R}$ be a sheaf of rings on X = Riemann surface.

- $\mathcal{R} = \mathcal{O}_X$ = sheaf of holomorphic functions
- $\mathcal{R} = \mathcal{C}^\infty$ = sheaf of smooth functions

Definition We say $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules if

- $\mathcal{F}(U)$ is an $\mathcal{R}(U)$-module $\forall\, U \subseteq X$ open
- $\forall\, U \supseteq V$ open, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{R}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

Remark We can speak about sheaves of $\mathcal{C}^{\infty}$-modules or sheaves of $\mathcal{O}_X$-modules.

Remark If $\mathcal{F}$ is a sheaf of $\mathcal{C}^{\infty}$-modules, it can be shown

$$H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$$

This fact uses the existence of partitions of unity & applies to a class of sheaves called *fine*.

We next consider sheaves of $\mathcal{O}_X$-modules, also called just $\mathcal{O}_X$-modules.

Example $\mathcal{O}_X(D)$ is $\mathcal{O}_X$-module for all divisors D.

Indeed, let $f \in \mathcal{O}_X(D)(U)$ and $g \in \mathcal{O}_X(U)$. We have

$$\operatorname{div} f + D|_U \geq 0.$$

We wish to show $gf \in \mathcal{O}_X(D)(U)$ that is

$$\operatorname{div}(gf) + D|_U \geq 0. \iff$$

$$\iff \underbrace{\operatorname{div} g}_{\geq 0} + \underbrace{(\operatorname{div} f + D|_U)}_{\geq 0} \geq 0$$ which is true since

g is holomorphic so $\operatorname{div} g \geq 0$.

Definition $\mathcal{F}$ is locally free of rank r provided

$\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ such that

$$\mathcal{F}/_{\mathcal{U}_\alpha} \cong \underbrace{\mathcal{O}_X/_{\mathcal{U}_\alpha} \oplus \cdots \oplus \mathcal{O}_X/_{\mathcal{U}_\alpha}}_{r \text{ copies}} \quad \text{as } \mathcal{O}_X/_{\mathcal{U}_\alpha}\text{-modules}$$

Remark $\mathcal{O}_X(D)$ is locally free of rank 1.

Indeed, let $X = \bigcup_\alpha \mathcal{U}_\alpha$ be coordinate charts. Write

$$D = \operatorname{div} f_\alpha \text{ in } \mathcal{U}_\alpha.$$

Then g is a section of $\mathcal{O}_X(D)/_{\mathcal{U}_\alpha}$ over $\mathcal{U} \subseteq \mathcal{U}_\alpha$ if

$$\operatorname{div} g + D/_{\mathcal{U}} \geq 0 \iff \operatorname{div} g + \operatorname{div} f_\alpha \geq 0 \text{ in } \mathcal{U}$$

$$\iff \operatorname{div} g f_\alpha \geq 0 \text{ in } \mathcal{U}$$

$$\iff g f_\alpha \in \mathcal{O}_X(\mathcal{U}).$$

Thus $\mathcal{O}_X(D)/_{\mathcal{U}_\alpha} \longrightarrow \mathcal{O}_X/_{\mathcal{U}_\alpha}$ is an isomorphism.

$$g \longrightarrow g f_\alpha$$

Remark

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}(x)$ is a module over $\mathcal{O}_X(x)$. In particular, $H^0(X, \mathcal{F}) = \mathcal{F}(x)$ is a $\mathbb{C}$-vector space

In fact $H^p(X, \mathcal{F})$ is a $\mathbb{C}$-vector space.

Coherent sheaves

$\mathcal{F} \longrightarrow X$ is coherent provided every point $x \in X$ admits a neighborhood $U \subseteq X$ and an exact sequence

$$\mathcal{O}_X\big|_U^{\oplus s} \longrightarrow \mathcal{O}_X\big|_U^{\oplus r} \longrightarrow \mathcal{F}\big|_U \longrightarrow 0$$

for some integers $r, s \geq 0$.

In particular $\mathcal{F} = \mathcal{O}_X(D)$ is coherent since it is locally free. (take $r = 1$, $s = 0$)