Math 220C - Lecture 5

April 6, 2022

Last time (Dirichlet Problem)

Given $f : \partial\Delta \longrightarrow \mathbb{R}$ continuous, define

$$u(re^{i\theta}) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})\, dt, & r < 1 \\ f(e^{i\theta}), & r = 1 \end{cases}$$

We have seen u harmonic in Δ & $u\big/_{\partial\Delta} = f$.

We show u continuous in $\bar{\Delta}$.

Conclusion u solves the Dirichlet Problem in $\Delta = \Delta(0,1)$.

Theorem $u : \overline{\Delta} \longrightarrow \mathbb{R}$ is continuous.

Proof The only issue is continuity over $\partial\Delta$ since u is continuous in Δ, being harmonic. We show

$$\lim_{\substack{r \to 1 \\ \theta \to \theta_0}} u(re^{i\theta}) = f(e^{i\theta_0}) \quad \forall\ \theta_0 .$$

Claim WLOG $\theta_0 = 0$

Else, rotate! Let

$\tilde{f}(z) = f(e^{i\theta_0} z)$. Let $\tilde{u}$ be the similar function with $\tilde{f}$ instead of f. By the explicit integral & change of variables

$$\tilde{u}(z) = u(e^{i\theta_0} z).$$

Thus u continuous at $\theta_0 \iff \tilde{u}$ is continuous at 1.

Let $\theta_0 = 0$ from now on.

Fix $\varepsilon > 0$. We show $\exists\, \rho, \delta > 0$ such that

$$|u(re^{i\theta}) - f(1)| < \varepsilon \quad \text{if} \quad |\theta| < \delta,\ \rho < r < 1$$

This completes the argument.

Since f is continuous, $\exists\ 0 < \delta < \pi$

(*) $$|f(e^{it}) - f(1)| < \varepsilon \quad \text{if} \quad |t| < \delta.$$

We let $M = \sup_{\partial\Delta} |f|$.

We estimate

$$|u(re^{i\theta}) - f(1)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})\, dt - \int_{-\pi}^{\pi} P_r(\theta - t)\, f(1)\, dt\right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, |f(e^{it}) - f(1)|\, dt \qquad (1)$$

We break this integral into two parts for $|t| \leq \delta$ and $|t| > \delta$.

$$\text{Term } \underline{I} = \frac{1}{2\pi} \int_{|t| \le \delta} P_r(\theta - t) \, |f(e^{it}) - f(1)| \, dt$$

$$\le \frac{1}{2\pi} \int_{|t| \le \delta} P_r(\theta - t) \cdot \varepsilon \, dt \quad \text{by } (*)$$

$$\le \frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) \, dt = \varepsilon \quad \text{by Lecture 4.} \qquad (2)$$

$$\text{Term } \underline{II} = \frac{1}{2\pi} \int_{|t| \ge \delta} P_r(\theta - t) \, |\underbrace{f(e^{it}) - f(1)}_{\le 2M}| \, dt$$

$$\le \frac{M}{\pi} \int_{|t| \ge \delta} P_r(\theta - t) \, dt$$

Claim If $|\theta| < \frac{\delta}{2} \implies P_r(\theta - t) \le P_r\left(\frac{\delta}{2}\right)$.

$$\implies \quad \text{Term } \underline{II} \le \frac{M}{\pi} \int_{|t| \ge \delta} P_r\left(\frac{\delta}{2}\right) dt$$

$$\le \frac{M}{\pi} \cdot P_r\left(\frac{\delta}{2}\right) \cdot 2\pi = 2M \, P_r\left(\frac{\delta}{2}\right) < \varepsilon. \qquad (3)$$

Indeed, $P_r\left(\frac{\delta}{2}\right) \to 0$ as $r \to 1$ by Lecture 4, so we can pick ρ such that $P_r\left(\frac{\delta}{2}\right) < \frac{\varepsilon}{2M}$ for $\rho < r < 1 \implies \text{Term } \underline{II} < \varepsilon$

Collecting (1), (2), (3) we find

$$|u(re^{i\theta}) - f(1)| < 2\varepsilon \quad \text{for} \quad |\theta| < \frac{\delta}{2}, \quad \rho < r < 1.$$

as needed.

Proof of the claim We may assume $t > 0$ so $t \in (\delta, \pi]$ while $\theta \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$. Then

$$t - \theta > \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

If $\pi \geq t - \theta$, since P_r is decreasing on $[\delta/2, \pi]$ (Lecture 4)

$$\Rightarrow P_r(\theta - t) = P_r(t - \theta) < P_r\left(\frac{\delta}{2}\right).$$

If $t - \theta > \pi \Rightarrow P_r(\theta - t) = P_r(2\pi - t + \theta)$. Note

$$\pi > 2\pi - t + \theta > 2\pi - \pi - \frac{\delta}{2} = \pi - \frac{\delta}{2} > \frac{\delta}{2}.$$

so again

$$P_r(\theta - t) = P_r(2\pi - t + \theta) < P_r\left(\frac{\delta}{2}\right).$$

Convolution Product

For functions $g, h : [-\pi, \pi] \longrightarrow \mathbb{R}$ continuous, 2π-periodic, set

$$g * h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta - t)\, h(t)\, dt.$$

If we write $u_r(\theta) = u(re^{i\theta})$ and write $f(t)$ instead of $f(e^{it})$,

we obtain

$$u_r = P_r * f.$$ Thus we defined the solution to the

Dirichlet problem as a convolution.

Heuristics

If g_r "converges to the δ-function" then heuristically

$$g_r * h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta - t)\, h(t)\, dt$$

$$\longrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\delta(\theta - t)}_{\text{concentrated at } \theta = t} h(t)\, dt = h(\theta).$$

The proof above makes this precise

Corollary The Dirichlet Problem can be solved in any disc $\Delta(a, R)$.

Why? This follows via translation & dilation

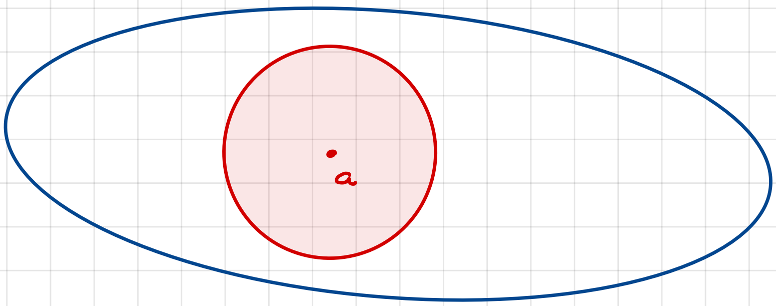
$$z \longrightarrow a + Rz.$$

mapping $\Delta(0,1) \longrightarrow \Delta(a, R)$. We solved the case of $\Delta(0,1)$ above.

Corollary (Converse to MVP)

If $u : G \longrightarrow \mathbb{R}$ continuous & satisfies MVP $\Rightarrow$ u harmonic

Proof



Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. We show u harmonic in $\Delta(a, R)$.

Let $f = u\big/_{\partial\Delta(a,R)}$. Solve Dirichlet Problem in $\bar{\Delta}(a,R)$.

Thus h harmonic in $\Delta(a, R)$, continuous in $\bar{\Delta}(a, R)$. & $h\big/_{\partial\Delta(a,R)} = f$.

Let $\Phi = h - u : \bar{\Delta}(a, R) \longrightarrow \mathbb{R}$. $\Rightarrow$ $\Phi\big/_{\partial\Delta(a,R)} = 0$ &

Φ continuous & satisfies MVP (because h, u do). Then $\Phi \equiv 0$

by Corollary to MP^+ (Lecture 2). Thus $u = h =$ harmonic in $\Delta(a, R)$.

Remark We say $u : G \longrightarrow \mathbb{R}$ continuous satisfies MVP locally if $\forall\, a \in G$ $\exists\, r > 0$, $\Delta(a, r) \subseteq G$ such that $\forall\, \rho < r$

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + \rho e^{it})\, dt.$$

The above proof shows

u satisfies MVP locally $\Longrightarrow$ u harmonic

Remark In consequence, if $u : G \longrightarrow \mathbb{R}$ continuous, TFAE

(i) u harmonic

(ii) u satisfies MVP

(iii) u satisfies MVP locally.