Math 220C - Lecture 6

April 8, 2022

§ 1. Convergence of harmonic functions   Conway X.2.

The natural notion of convergence for harmonic functions is local uniform convergence.

Recall: Let $f_n, f : G \longrightarrow \mathbb{R}$

(1) local uniform convergence $f_n \overset{l.u.}{\rightrightarrows} f$ means

$\forall x \in G \ \exists U_x \subseteq G$ open near x, such that

$f_n \rightrightarrows f$ uniformly in U_x

(2) uniform convergence on compact sets $f_n \overset{c}{\rightrightarrows} f$

means $\forall K$ compact, $K \subseteq G$, $f_n \rightrightarrows f$ uniformly in K.

Recall: The two notions agree.

Indeed (1) $\Rightarrow$ (2) because any compact set can be covered by finitely many U_x's.

Lemma

If $u_n : G \longrightarrow \mathbb{R}$ harmonic & $u_n \overset{l.u.}{\rightrightarrows} u$ then $u : G \rightarrow \mathbb{R}$ harmonic.

Proof Since u_n harmonic $\Rightarrow$ u_n continuous $\Rightarrow$ u continuous.

Since u_n harmonic $\Rightarrow$ u_n satisfies MVP. Let $\overline{\Delta}(a,R) \subseteq G$.

$$u_n(a) = \frac{1}{2\pi} \int_0^{2\pi} u_n(a + Re^{it})\, dt$$

Make $n \rightarrow \infty$.

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it})\, dt$$

$\Rightarrow$ u satisfies MVP. $\Rightarrow$ u harmonic.

§ 2 Harnack's Theorem

Let $u_n : G \longrightarrow \mathbb{R}$ harmonic and

$u_1 \leq u_2 \leq \ldots \leq u_n \leq \ldots$ in G. Then either

(1) $u_n \overset{\ell.u.}{\rightrightarrows} u$ & u harmonic. or

(2) $u_n \overset{\ell.u.}{\rightrightarrows} \infty$.

Remark If $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$ are real numbers,

then either

(1) $a_n \longrightarrow a < \infty$

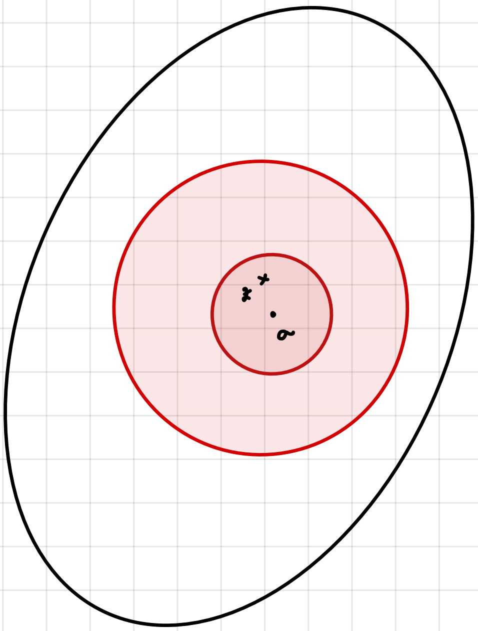
(2) $a_n \longrightarrow \infty$

Remark Harnack Inequality (Lecture 3)

$v : G \longrightarrow \mathbb{R}$, $v \geq 0$, harmonic, $\overline{\Delta}(a, R) \subseteq G$.

If $|z - a| = r < R$, then

$$v(a)\,\frac{R-r}{R+r} \leq v(z) \leq v(a)\,\frac{R+r}{R-r}$$



If $r \leq R/2$. $\Rightarrow$ $\frac{R+r}{R-r} \leq 3$, $\frac{1}{3} \leq \frac{R-r}{R+r}$.

If $z \in \Delta\left(a, \frac{R}{2}\right)$ then $\frac{1}{3}\,v(a) \leq v(z) \leq 3\,v(a)$.

Proof of Harnack's theorem

WLOG $u_n \geq 0$. else work with $\tilde{u}_n = u_n - u_1 \geq 0$

Step 1 Pointwise convergence.

Since $\{u_n(z)\}$ is non decreasing $\forall z \in G \Rightarrow$

$\Rightarrow$ either $u_n(z) \longrightarrow \infty$ or $u_n(z) \longrightarrow u(z)$ for some $u(z) < \infty$.

Let $A = \{z \in G : u_n(z) \longrightarrow \infty\}$ $\Rightarrow A \cap B = \emptyset$, $A \cup B = G$.

$B = \{z \in G : u_n(z) \longrightarrow u(z)\}$

It suffices to show A, B open. Since G connected $\Rightarrow$

$A = G$ or $B = G$.

Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. Let $z \in \bar{\Delta}(a, \frac{R}{2})$.

$$\Rightarrow \frac{1}{3} u_n(a) \leq u_n(z) \leq 3\, u_n(a).$$

$\boxed{i}$ If $a \in A \Rightarrow u_n(a) \longrightarrow \infty. \Rightarrow u_n(z) \longrightarrow \infty. \Rightarrow z \in A$

$$\Rightarrow \Delta\left(a, \frac{R}{2}\right) \subseteq A. \Rightarrow A \text{ open}$$

$\boxed{ii}$ If $a \in B \Rightarrow u_n(a) \longrightarrow u(a) < \infty. \Rightarrow u_n(z) \longrightarrow u(z) < \infty \Rightarrow z \in B.$

$$\Rightarrow \Delta\left(a, \frac{R}{2}\right) \subseteq B. \Rightarrow B \text{ open}.$$

Step 2 Local uniform convergence

Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. We show uniform convergence in $\Delta(a, \frac{R}{2})$. We have two cases:

$\boxed{i}$ $u_n(a) \longrightarrow \infty$. $\Rightarrow \forall M \ \exists N : u_n(a) \geq 3M$ for $n \geq N$

$$\Rightarrow u_n(z) \geq \frac{1}{3} u_n(a) \geq M \quad \forall n \geq N, \ z \in \Delta\left(a, \tfrac{R}{2}\right).$$

$$\Rightarrow u_n \rightrightarrows \infty \quad \text{in} \quad \Delta\left(a, \tfrac{R}{2}\right).$$

$\boxed{ii}$ $u_n(a) \longrightarrow u(a)$. Fix $\varepsilon > 0$. Since $\{u_n(a)\}$ Cauchy

$$\Rightarrow \exists N : 0 \leq u_n(a) - u_m(a) < \frac{\varepsilon}{3} \quad \forall n \geq m \geq N$$

$$\Rightarrow 0 \leq u_n(z) - u_m(z) < 3\,(u_n(a) - u_m(a)) < \varepsilon \quad \forall n \geq m \geq N.$$

Make $n \longrightarrow \infty$ $\Rightarrow$ $0 \leq u(z) - u_m(z) \leq \varepsilon \quad \forall m \geq N, \ z \in \Delta\left(a, \frac{R}{2}\right).$

$$\Rightarrow u_m \rightrightarrows u \quad \text{in} \quad \Delta\left(a, \tfrac{R}{2}\right).$$

2. Subharmonic Functions   Conway $\overline{X}$. 3.

SH functions share many properties with harmonic fns.

Definition $\varphi : G \longrightarrow \mathbb{R}$ continuous, $\forall\, a \in G$, $\exists\ \overline{\Delta}(a, R) \subseteq G$.

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + r e^{it})\, dt \qquad \forall\ 0 \leq r \leq R$$

then φ is called subharmonic.

Superharmonic functions satisfy the opposite inequality.

Remark

[i] φ subharmonic $\Rightarrow -\varphi$ superharmonic

[ii] φ harmonic $\Rightarrow \varphi$ sub/superharmonic

[iii] φ is C^2 & $\Delta \varphi \geq 0 \Rightarrow \varphi$ subharmonic.

This is HWK 2, Problem 1.

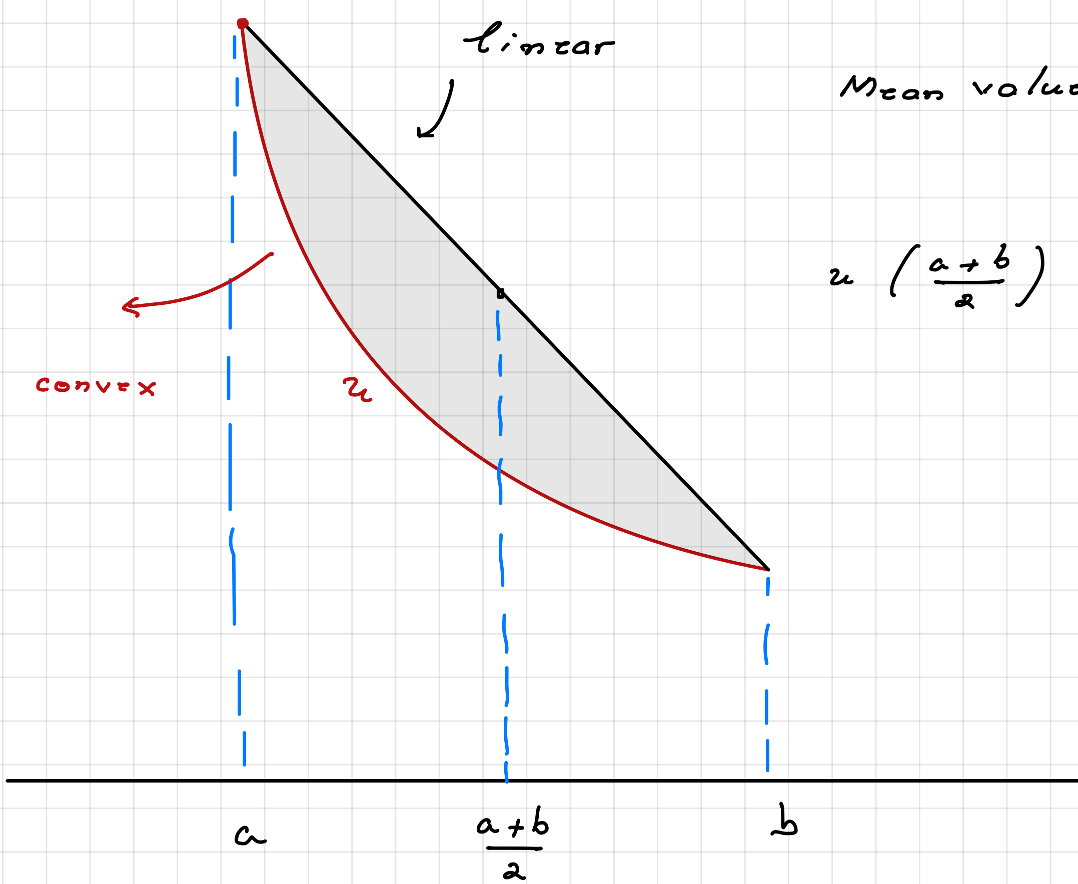
Analogy with 1 real variable

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \longleftrightarrow \frac{\partial^2}{\partial x^2} \quad \text{(1 variable)}$$

"harmonic" $\quad \dfrac{\partial^2 u}{\partial x^2} = 0 \implies u$ linear, $u = ax + b$.

C^2 "subharmonic" $\quad \dfrac{\partial^2 u}{\partial x^2} \geq 0 \implies u$ convex

C^2 "superharmonic" $\quad \dfrac{\partial^2 u}{\partial x^2} \leq 0 \implies u$ concave



Mean value Inequality becomes

$$u\left(\frac{a+b}{2}\right) \leq \frac{u(a) + u(b)}{2}$$