Math 220C - Lecture 7

April 11, 2022

Definition $\varphi : G \longrightarrow \mathbb{R}$ continuous, $\forall a \in G$, $\exists \overline{\Delta}(a, R) \subseteq G$.

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + r e^{it})\, dt \qquad \forall\ 0 \leq r \leq R$$

then φ is called subharmonic.

Properties of subharmonic functions

(1) similar to harmonic functions

(2) new properties

(1) Maximum principle (old property)

$\varphi : G \longrightarrow \mathbb{R}$ subharmonic

MP: If $\varphi : G \longrightarrow \mathbb{R}$ SH and achieves a maximum at $a \in G$

$\Rightarrow \varphi$ constant.

MP^+: If $\varphi : G \longrightarrow \mathbb{R}$ SH and $\forall\, a \in \partial_\infty G$,

$$\limsup_{z \to a} \varphi(z) \leq 0 \Rightarrow \varphi < 0 \text{ or } \varphi \equiv 0 \text{ in } G.$$

Corollary $\varphi : \overline{G} \longrightarrow \mathbb{R}$, G bounded, φ continuous in $\overline{G}$, subharmonic in G, $\varphi/_{\partial G} \leq 0$. Then $\varphi < 0$ or $\varphi \equiv 0$ in G.

(2) New Property

φ_1, φ_2 subharmonic $\Rightarrow$ $\varphi = \max(\varphi_1, \varphi_2)$ subharmonic.

Proof φ continuous. If $a \in G$, we can find R_1, R_2 with

$$\varphi_1(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi_1(a+re^{it})\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt \quad \forall\, r < R_1$$

$$\varphi_2(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi_2(a+re^{it})\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt \quad \forall\, r < R_2$$

For $R = \min(R_1, R_2)$, $r < R$ we have

$$\varphi(a) = \max(\varphi_1(a), \varphi_2(a)) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt$$

$\Rightarrow$ φ subharmonic.

New Property (Poisson Modification / Bumping)

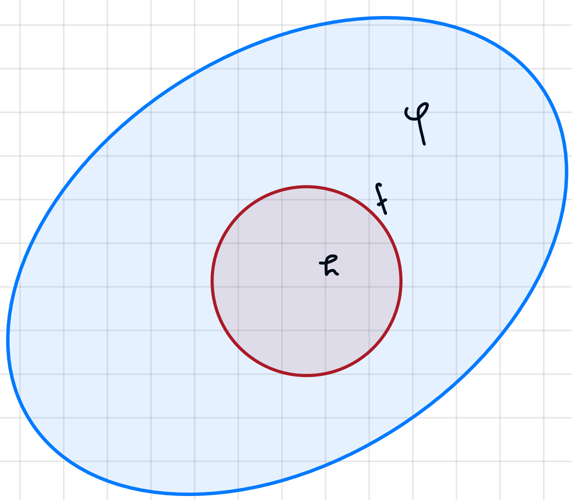
φ subharmonic $\Rightarrow$ $\tilde{\varphi}$ subharmonic

Construction (Poisson modification / Bumping)

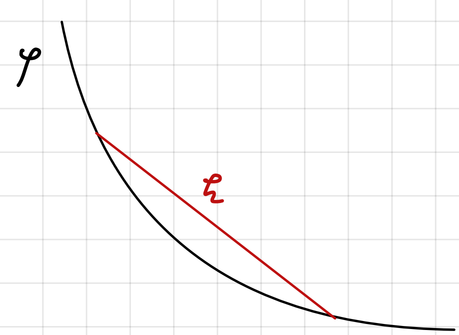
- $\varphi : G \longrightarrow \mathbb{R}$ subharmonic
- $\overline{\Delta} \subseteq G$ closed disc
- $f = \varphi|_{\partial\Delta}$.
- Solve Dirichlet Problem in $\overline{\Delta}$:

  h continuous in $\overline{\Delta}$, harmonic in Δ, $h|_{\partial\Delta} = f$.

- Let $\tilde{\varphi} = \begin{cases} \varphi & \text{in } G \setminus \overline{\Delta} \\ h & \text{in } \overline{\Delta} \end{cases}$ $\Rightarrow$ $\tilde{\varphi}$ cont.



in $\mathbb{R}^2$



in $\mathbb{R}$.

Proposition Conway 3.7$^+$

[i] $\varphi \le \tilde{\varphi}$

[ii] $\tilde{\varphi}$ subharmonic (HWK 3)

[iii] $\varphi_1 \le \varphi_2$ subharmonic $\Longrightarrow$ $\tilde{\varphi}_1 \le \tilde{\varphi}_2$.

Proof [i] Since $\varphi = \tilde{\varphi}$ in $G \setminus \bar{\Delta}$, we only need to prove

$$\varphi \le h \text{ in } \bar{\Delta}.$$

Note that $\varphi - h$ is subharmonic (φ satisfies MV-inequality, h satisfies MV-equality). Note

$$\varphi - h \big|_{\partial\Delta} = f - f = 0.$$

By Maximum Principle $\varphi - h \le 0$ in $\bar{\Delta}$, as needed.

$\boxed{iii}$ Let $f_1 = \varphi_1\big/_{\partial\Delta}$, $f_2 = \varphi_2\big/_{\partial\Delta}$.

Let $h_1, h_2 : \overline{\Delta} \longrightarrow \mathbb{R}$ solve Dirichlet Problem with boundary values f_1, f_2.

To show $\tilde{\varphi}_1 \le \tilde{\varphi}_2$, it suffices to show $h_1 \le h_2$ in $\overline{\Delta}$.

Note $h_1 - h_2$ harmonic in Δ, continuous in $\overline{\Delta}$

$$h_1 - h_2\big/_{\partial\Delta} = f_1 - f_2 = \varphi_1\big/_{\partial\Delta} - \varphi_2\big/_{\partial\Delta} \le 0$$

By Maximum Principle, $h_1 - h_2 \le 0$ in $\overline{\Delta}$ as needed.

Question How do we construct interesting harmonic functions?

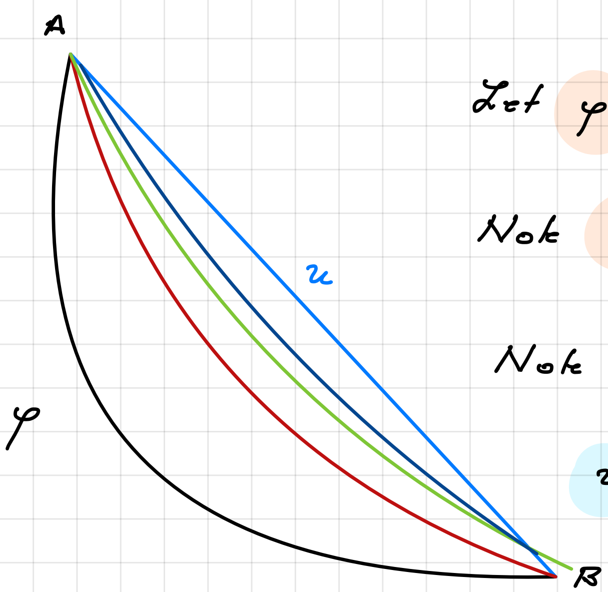
Methods

$\boxed{i}$ $u = \mathrm{Re}\, f$, f holomorphic

$\boxed{ii}$ Poisson's formula / Dirichlet Problem, $G = \Delta$

$\boxed{iii}$ Perron method

Idea behind Perron's method – 1 variable



Let φ be a convex path joining A, B.

Note $\varphi \le u$ where u = line.

Note

$$u(z) = \sup\{\varphi(z) : \varphi \text{ convex as above}\}.$$

We wish to extend these observations to $\mathbb{R}^2$.

In $\mathbb{R}^2$: $G \subseteq \mathbb{C}$ bounded, $f : \partial G \longrightarrow \mathbb{R}$ continuous

- Perron family

$$\mathcal{P}(G, f) = \left\{ \varphi : G \longrightarrow \mathbb{R} \text{ subharmonic}, \ \limsup_{z \to a} \varphi(z) \le f(a) \ \forall \ a \in \partial G. \right\}.$$

- Perron function $u : G \longrightarrow \mathbb{R}$

$$u(z) = \sup \left\{ \varphi(z), \ \varphi \in \mathcal{P}(G, f) \right\}$$

Question: Is the Perron function well-defined?

Remarks $\boxed{I}$ $\mathcal{P}(G, f) \neq \emptyset$.

Indeed, ∂G compact, f cont. $\Longrightarrow$ $m \leq f \leq M$ in ∂G.

Then $\varphi = m$ is in $\mathcal{P}(G, f)$.

$\boxed{II}$ u is well-defined.

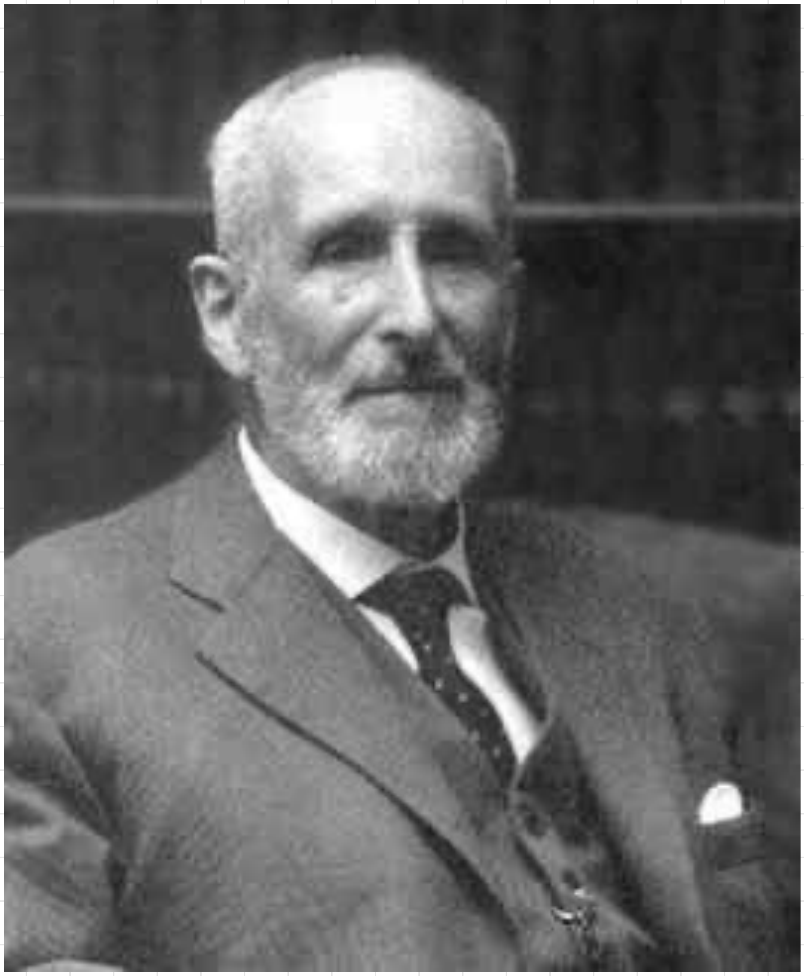
Since $f \leq M$, we have

$$\limsup_{z \to a} \varphi(z) \leq M \quad \forall\, a \in \partial G \Longrightarrow \varphi \leq M \text{ by MP}$$

$$\Longrightarrow u(z) = \sup\{\varphi(z)\} \leq M.$$

Theorem Conway 3.11.

The Perron function u is harmonic



Oskar Perron (1880 – 1975) was a German mathematician. He brought contributions to PDE's, known for the Perron method.