

Math 220C - Lecture 8

April 18, 2022

$G \subseteq \mathbb{C}$ bounded, $f: \partial G \rightarrow \mathbb{R}$ continuous

• Perron family

$$\mathcal{P}(G, f) = \left\{ \varphi: G \rightarrow \mathbb{R} \text{ subharmonic, } \limsup_{z \rightarrow a} \varphi(z) \leq f(a) \forall a \in \partial G. \right\}$$

• Perron function $u: G \rightarrow \mathbb{R}$

$$u(z) = \sup \{ \varphi(z), \varphi \in \mathcal{P}(G, f) \}$$

Remark

$$\varphi \in \mathcal{P}(G, f) \Rightarrow \tilde{\varphi} \in \mathcal{P}(G, f)$$

Indeed, $\tilde{\varphi}$ subharmonic and $\varphi = \tilde{\varphi}$ near $a \in \partial G$ so

$$\limsup_{z \rightarrow a} \tilde{\varphi}(z) = \limsup_{z \rightarrow a} \varphi(z) \leq f(a).$$

Remark

$\varphi, \psi \in \mathcal{P}(G, f)$ then $\max(\varphi, \psi) \in \mathcal{P}(G, f)$

Theorem Conway 3.11.

The Perron function u is harmonic

Proof Let $x \in G$, $\bar{\Delta} \subseteq G$ a disc around x .

WTS u harmonic in Δ .

Step 1 Find functions $\varphi_n \in \mathcal{P}(G, f)$ with $\varphi_n(x) \rightarrow u(x)$.

This is possible by the definition of u .

WLOG we may assume $\varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_n \leq \dots$

Why? Else, define

$$\varphi_n^{\text{new}} = \max(\varphi_1, \varphi_2, \dots, \varphi_n).$$

By Lecture 6, Property (2), $\varphi_n^{\text{new}} \in \mathcal{P}(G, f)$. Note that

$\varphi_n^{\text{new}}(x) \rightarrow u(x)$ as well and that

$$\varphi_1^{\text{new}} \leq \varphi_2^{\text{new}} \leq \dots \leq \varphi_n^{\text{new}} \leq \dots$$

We drop the superscript "new" from now on.

Step 2 WLOG We may assume

$$\varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_n \leq \dots \quad \text{are harmonic in } \Delta.$$

Why? Indeed,

$$\tilde{\varphi}_1 \leq \tilde{\varphi}_2 \leq \dots \leq \tilde{\varphi}_n \leq \dots \quad (\text{see Proposition above})$$

and $\tilde{\varphi}_n \in \mathcal{P}(G, f)$ by Remark III. Note $\tilde{\varphi}_n$ are harmonic in Δ .

Furthermore $\tilde{\varphi}_n(x) \rightarrow u(x)$ still holds. Indeed,

Proposition II definition of u as supremum

$$\varphi_n(x) \leq \tilde{\varphi}_n(x) \leq u(x)$$

Thus $\varphi_n(x) \rightarrow u(x) \Rightarrow \tilde{\varphi}_n(x) \rightarrow u(x)$

We can work with the functions $\tilde{\varphi}_n$ instead of the φ_n 's.

Step 3 By Harnack's convergence

$$\tilde{\varphi}_n \xrightarrow{\text{l.u.}} U \text{ in } \Delta, \text{ for } U \text{ harmonic.}$$

We noted that $\tilde{\varphi}_n(x) \rightarrow u(x) < \infty$ so the possibility

$\tilde{\varphi}_n \xrightarrow{\text{l.u.}} \infty$ in Harnack is not allowed.

Note $U(x) = u(x)$.

Goal

We show $U = u$ in Δ . (not only at x).

This will show u is harmonic, as needed

Step 4 Let $y \in \Delta$. We show $\mathcal{U}(y) = u(y)$.

Let $\varphi_n \in \mathcal{P}$, $\varphi_n(y) \rightarrow u(y)$, possible by definition of u .

WLOG $\varphi_n \leq \psi_n$

Why? $\varphi_n^{\text{new}} = \max(\varphi_n, \psi_n) = \text{subharmonic (Lecture 6)}$.

We still have $\varphi_n^{\text{new}} \in \mathcal{P}(G, f)$.

Bonus $\varphi_n^{\text{new}}(x) \rightarrow u(x)$ & $\varphi_n^{\text{new}}(y) \rightarrow u(y)$.

Why?

We know $\varphi_n(x) \rightarrow u(x)$
 $\varphi_n \leq \varphi_n^{\text{new}} \leq u$ $\implies \varphi_n^{\text{new}}(x) \rightarrow u(x)$.
def. of φ_n^{new} \rightsquigarrow definition of u as supremum

The same argument works for y , with ψ_n instead of φ_n .

We run the above *Steps* for ψ_1, ψ_2, \dots

Step 1' $\psi_1 \leq \psi_2 \leq \dots \leq \psi_n \leq \dots$

Step 2' $\tilde{\psi}_1 \leq \tilde{\psi}_2 \leq \dots \leq \tilde{\psi}_n \leq \dots$ harmonic in Δ

Step 3' Harnack $\tilde{\psi}_n \xrightarrow{p.u.} V$ in Δ .

Claims

[a] $V(y) \stackrel{\text{Step 3}'}{=} \lim_{n \rightarrow \infty} \tilde{\psi}_n(y) \stackrel{\text{Bonus}}{=} u(y)$

[b] $V(x) \stackrel{\text{Step 3}'}{=} \lim_{n \rightarrow \infty} \tilde{\psi}_n(x) \stackrel{\text{Bonus}}{=} u(x)$

[c] $U(z) \stackrel{\text{Step 3}}{=} \lim \tilde{\varphi}_n(z) \leq \lim \tilde{\psi}_n(z) \stackrel{\text{Step 3}'}{=} V(z)$

using that $\varphi_n \leq \psi_n$ and $\tilde{\varphi}_n \leq \tilde{\psi}_n$.

Conclusion

We know $U - V \leq 0$ in Δ by [c]

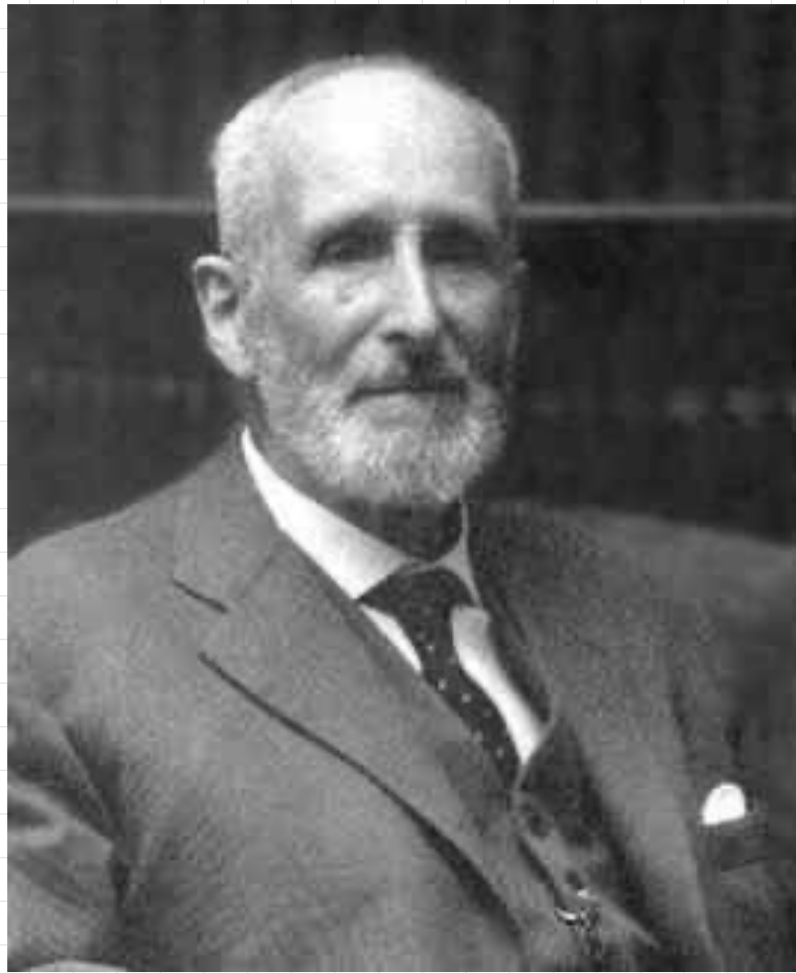
$$U(x) = u(x) = V(x) \Rightarrow (U - \bar{V})(x) = 0.$$

\downarrow Steps \downarrow [b]

By Max. Principle $\Rightarrow U - V \equiv 0$. \swarrow harmonic

In particular, $U(y) = V(y) = u(y)$. \swarrow [a]

Since $y \in \Delta$ is arbitrary $\Rightarrow U \equiv u$, as needed.



Oskar Perron (1880 - 1975) was a German mathematician. He brought contributions to PDE's, known for the Perron method.

Question Does the Perron function solve Dirichlet Problem?

What is the issue?

We know u is harmonic in G .

We need to show $\lim_{z \rightarrow a} u(z) = f(a) \quad \forall a \in \partial G$.

Answer (HWK 3, #4) NO!

If $G = \Delta(0,1) \setminus \{0\}$, we show that the Dirichlet Problem does not always admit a solution.

Better answer In special cases, it does!