Math 220C - Lecture 9

April 15, 2022

Last time

Question Does the Perron function solve the Dirichlet Problem?

Answer (HWK 3, #4) **NO!** Take $G = \Delta \setminus \{0\}$.

Better answer In special cases, it does!

Terminology (differs from Conway $\overline{X}$. 4)

Let G be bounded. Let $a \in \partial G$.

$\omega : \overline{G} \longrightarrow \mathbb{R}$ continuous in $\overline{G}$, harmonic in G,

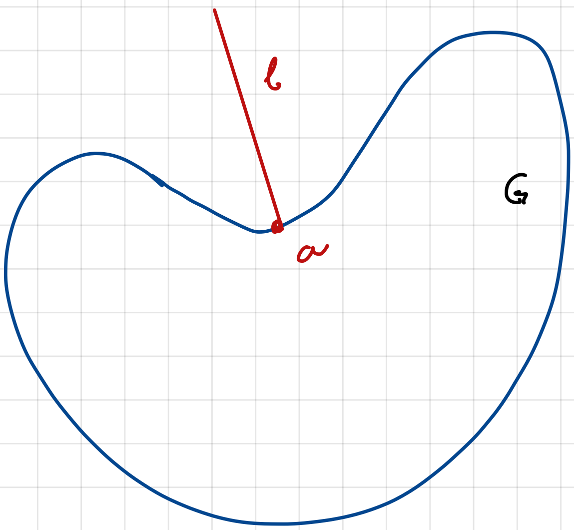
$\omega(a) = 0, \quad \omega > 0$ in $\partial G \setminus \{a\}$

ω is said to be a **barrier** at a.

By the minimum principle $\Rightarrow$ $\omega > 0$ in $\overline{G} \setminus \{a\}$

The terminology is due to Lebesgue.

Example (HWK 3, #6) Many reasonable domains satisfy this definition. For instance, if $\exists$ ℓ segment $\ell \cap \overline{G} = \{a\}$ then there is a barrier at a.



Example $G = \Delta \setminus \{0\}$. In this case, there is no barrier at 0. Indeed, if $\omega(0) = 0$ but $\omega > 0$ on $\partial G \setminus \{0\}$ we contradict the mean value property.

Remark If the Dirichlet Problem is solvable in G $\Rightarrow$ $\forall\, a \in \partial G$ admits a barrier. (HWK 3, #5).

Conversely

Let G be bounded and assume each $a \in \partial G$ has a barrier. Let $f : \partial G \longrightarrow \mathbb{R}$ be continuous.

Theorem The Perron function u for (G, f) satisfies

$$\lim_{z \to a} u(z) = f(a)$$

Corollary The Perron function solves the Dirichlet Problem under the above assumptions.

We let ω be a barrier at a. Thus

- $\omega : \overline{G} \longrightarrow \mathbb{R}$, ω cont in $\overline{G}$, ω harmonic in G
- $\omega(a) = 0$, $\omega > 0$ in $\partial G \setminus \{a\}$.

Proof WLOG $f(a)=0$. Let $\varepsilon>0$. We show

[i] $\displaystyle\limsup_{z\to a} u(z) \le \varepsilon$

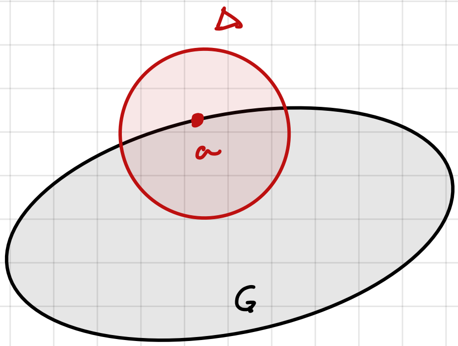
[ii] $\displaystyle\liminf_{z\to a} u(z) \ge -\varepsilon$

Then $\displaystyle\lim_{z\to a} u(z) = 0 = f(a)$, as needed.

Let Δ be a disc with

$$-\varepsilon < f < \varepsilon \quad \text{in } \partial G \cap \Delta \text{ and } a \in \Delta. \qquad (1)$$

Let $M = \underset{\partial G\,=\,\text{compact}}{\sup} |f| \Rightarrow -M \le f \le M$ in ∂G. (2)



Since $\bar{G} \setminus \Delta$ is compact, let

$$\omega_0 = \min_{\bar{G}\setminus\Delta} \omega > 0$$

Why? By Minimum Principle^{+}, either $\omega \equiv 0$ in G (not true as $\omega|_{\partial G} \not\equiv 0$) or else $\omega > 0$ in G. But $\omega > 0$ in $\partial G \setminus \{a\}$. Thus $\omega > 0$ in $\bar{G}\setminus\{a\}$. Since $\bar{G}\setminus\Delta \subseteq \bar{G}\setminus\{a\}$, we get the claim.

Proof of $\boxed{II}$

Let $V(z) = -\varepsilon - \frac{\omega(z)}{\omega_0} \cdot M.$ = harmonic in G.

cont in $\bar{G}$

Claim 1 $V \leq f$ over ∂G

Proof Let $z \in \partial G$.

$\omega \geq 0$ on ∂G .

- $z \in \partial G \cap \Delta$: $V(z) \leq -\varepsilon \overset{(1)}{<} f(z)$
- $z \in \partial G \setminus \Delta$: $V(z) < -M \overset{(2)}{\leq} f(z)$

$\omega \geq \omega_0$ in $\bar{G} \setminus \Delta$

Claim 2 $V \in \mathcal{P}(G, f)$.

Proof We know V harmonic. For $\zeta \in \partial G$,

$$\lim_{z \to \zeta} V(z) = V(\zeta) \leq f(\zeta) \text{ by Claim 1.}$$

Since u is defined as a supremum over $\mathcal{P}(G, f)$ & $V \in \mathcal{P}(G,f)$

$\Rightarrow$ $u(z) \geq V(z) \quad \forall z \in G$

$\Rightarrow$ $\liminf_{z \to a} u(z) \geq V(a) = -\varepsilon$ as needed.

$\hookrightarrow \omega(a) = 0.$

Proof of $\boxed{\mathrm{I}}$ Let

$$W(z) = \varepsilon + \frac{\omega(z)}{\omega_0} \cdot M = \text{harmonic in } G, \text{ cont. in } \bar{G}.$$

Claim 1' $W \geq f$ over ∂G.

Proof

$\omega \geq 0$ in ∂G

- $z \in \partial G \cap \Delta$, $W(z) \geq \varepsilon \overset{(1)}{>} f(z)$

- $z \in \partial G \setminus \Delta$, $W(z) > M \overset{(2)}{\geq} f(z)$

$\omega \geq \omega_0$ in $\bar{G} \setminus \Delta$

We do not know $W \in \mathcal{P}$, but we can compare W to any $\varphi \in \mathcal{P}$

Claim 2' $W(z) \geq \varphi(z)$ $\forall \varphi \in \mathcal{P}$ $\forall z \in G$.

Proof Let $\zeta \in \partial G$. Then

definition of $\mathcal{P}$ claim 1'

$$\limsup_{z \to \zeta} \varphi(z) \leq f(\zeta) < W(\zeta) = \lim_{z \to \zeta} W(z)$$

$\Rightarrow \varphi(z) \leq W(z)$ $\forall z \in G$ by MP^+ applied to the function $\varphi - W$.

Since $u(z) = \sup\{\varphi(z) : \varphi \in \mathcal{P}\} \Rightarrow u(z) \le W(z)$ by

Claim 2 $\forall z \in \mathbb{C}$. Then

$$\limsup_{z \to a} u(z) \le \lim_{z \to a} W(z) = W(a) = \varepsilon,$$ as needed.

Remark The proof only used that ω is superharmonic.

& some references require barriers to be superharmonic.

(versus harmonic).

Remark Let $a \in \partial G$, $G(a) = G \cap \Delta(a, R)$, a neighborhood of a.

We say $\omega : \overline{G(a)} \longrightarrow \mathbb{R}$ is a *local barrier* at a if

[i] ω is continuous, ω superharmonic in $G(a)$

[ii] $\omega(a) = 0$, $\omega > 0$ on $\overline{G(a)} \setminus \{a\}$.

A-priori, the notion of a local barrier is more flexible.

However, a local barrier exists $\Longleftrightarrow$ global barrier exists.

Indeed, let $\overline{\Delta}'(a, r) \subseteq \Delta(a, R)$, let $\omega_0 = \inf\limits_{\overline{G(a)} \setminus \Delta'} \omega$. Set

$$\hat{\omega}(z) = \begin{cases} \min(\omega(z), \omega_0) & : z \in \overline{G} \cap \Delta' \\ \omega_0 & : z \in \overline{G} \setminus \Delta' \end{cases}$$

$\Rightarrow$ $\hat{\omega}$ is a global barrier.

Thus for a bounded $G \subseteq \mathbb{C}$, we have

G is Dirichlet region $\Longleftrightarrow$ local barriers exist at all $a \in \partial G$.

Remark It is shown in Conway X. 4.9 that if a bounded

$G \subseteq \mathbb{C}$ satisfies

> $\hat{\mathbb{C}} \setminus G$ has no connected components which reduce to a point

then G has a barrier at each $a \in \partial G$

Thus the Dirichlet Problem can be solved in such a region.