Math 220, Problem Set 2.

1. (Applications of the $\Gamma$-function. Wednesday, January 13.) Let 
   \[ R(z) = \frac{P(z)}{Q(z)} \]
   be a rational function without poles at the positive integers.

   **Remark:** To put things into perspective, recall that Problem 5 on the Final Exam for Math 220A gave a formula for the infinite sum $\sum_{k=-\infty}^{\infty} R(k)$ under certain assumptions. In this problem, we study the infinite product $\prod_{k=1}^{\infty} R(k)$.

   Assume that $P, Q$ are polynomials of the same degree, with leading term equal to 1. Write 
   \[ P(z) = \prod_{i=1}^{d} (z - a_i), \quad Q(z) = \prod_{i=1}^{d} (z - b_i) \]
   and furthermore assume that 
   \[ \sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i. \]
   (One can see that without these assumptions the product does not converge.)

   (i) Using the definition of the function $G$, show that
   \[ \prod_{k=1}^{\infty} R(k) = G(-a_1) \cdots G(-a_d) \frac{G(-b_1) \cdots G(-b_d)}{G(\alpha)}. \]
   Express the product $\prod_{k=1}^{\infty} R(k)$ in terms of the $\Gamma$ function.

   (ii) Using (i), compute the product
   \[ \prod_{n=1}^{\infty} \frac{n^2 + n - 4/9}{n^2 + n - 5/16}. \]
   You can simplify the answer so that only sine’s are involved.

2. (Factorization of trigonometric functions. Monday, January 11.) This is a version of Problem 3, Conway VII.6 with some hints.

   Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Show that
   \[ \frac{\sin \pi (z + \alpha)}{\sin \pi \alpha} = e^{\pi z \cot \pi \alpha} \prod_{n=-\infty}^{\infty} \left( 1 - \frac{z}{n - \alpha} \right) e^{\pi z \cot \pi \alpha} \prod_{n=-\infty}^{\infty} E_1 \left( \frac{z}{n - \alpha} \right). \]
   Use this to find a factorization of the function
   \[ \cos \left( \frac{\pi}{4} z \right) - \sin \left( \frac{\pi}{4} z \right). \]
   **Hint:** You will need the usual strategy given in class: examine the zeros and take logarithmic derivatives. Also recall the identities of Problem Set 6 from Math 220A.
For the next two questions, you will only need to know that the Weierstraβ problem admits a solution. (Feel free to use the explicit form of the solution if it helps you, though this is not strictly speaking needed.)

3. (“Greatest common divisor.” Friday, January 15.) This is Conway VII.5, Problem 3. Assume that \( f \) and \( g \) are entire functions. Show that there exist entire functions \( h, F \) and \( G \) such that
\[
f(z) = h(z)F(z), \quad g(z) = h(z)G(z)
\]
with \( F, G \) having no common zeroes.

4. (Roots. Friday, January 15.) Let \( f \) be an entire function and \( n \geq 1 \). Show that there exists an entire function \( g \) such that \( g^n = f \) if and only if the orders of all zeroes of \( f \) are divisible by \( n \).

The final question introduces a new entire function that was not part of the traditional arsenal of examples in Math 220A.

5. (The Weierstraβ σ-function. Friday, January 15.) This is a modified version of Conway VII.5, Problem 13.

Let \( \omega_1, \omega_2 \) be two non-zero complex numbers such that \( \omega_2/\omega_1 \not\in \mathbb{R} \). Let
\[
\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.
\]
Solve the Weierstraβ problem of finding entire functions with simple zeroes at the lattice points in \( \Lambda \).

To this end, define the Weierstraβ σ-function as the infinite product
\[
\sigma(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \exp \left(\frac{z}{\lambda} + \frac{1}{2} \cdot \frac{z^2}{\lambda^2}\right) = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2 \left(\frac{z}{\lambda}\right).
\]

(i) Show that \( \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3} \) converges.

\textit{Hint:} Show that there exists a number \( c > 0 \) such that
\[
|n\omega_1 + m\omega_2| \geq c(|n| + |m|),
\]
for all real numbers \( n, m \). Show that the number of integer solutions of \( |n| + |m| = k \) is equal to \( 4k \).

Conclude that
\[
\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3} \leq 4c^{-3} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]

(ii) Show that \( \sigma \) is an entire function with simple zeroes only at the points of \( \Lambda \).

\textit{Remark:} The \( \sigma \) function is important in the study of elliptic functions and the study of Riemann surfaces of genus 2.