Last time

Arzelà–Ascoli Terminology

Today — we give the proof.

All functions today are continuous.
**Notation & Preliminaries**

$f : U \rightarrow \mathbb{R}$ continuous, $K \subseteq U$ compact

$$\|f\|_K = \sup_{x \in K} |f(x)|$$

**Note**

1. $\|f + g\|_K \leq \|f\|_K + \|g\|_K$

ii. $f_n \to f \iff \|f_n - f\|_K \to 0$ as $n \to \infty$.

**Def**

$f_n$ is uniformly Cauchy in $K$ if

$$\forall \varepsilon > 0 \exists N \forall n, m \geq N, \quad \|f_n - f_m\|_K < \varepsilon.$$
Lemma \( f_n \) converges uniformly in \( K \)

\[ \iff \quad f_n \text{ uniformly Cauchy in } K. \]

Proof We will only use \( \Rightarrow \) so we only give its proof.

Fix \( \varepsilon > 0 \) \( \Rightarrow \) \( \exists N \) with \( |f_n(x) - f_m(x)| < \varepsilon \) \( \forall n, m \geq N. \)

\[ \forall x \in K. \quad (\ast) \]

Thus \( \{ f_n(x) \} \) is Cauchy for fixed \( x \). Then \( \{ f_n(x) \} \) converges pointwise to \( f(x) \). Make \( m \rightarrow \infty \) in \((\ast)\) to conclude that

\[ \forall \varepsilon > 0 \quad \exists N \quad \text{with} \quad |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, \quad x \in K. \]

Thus \( f_n \xrightarrow{\text{pointwise}} f \) in \( K \).
Proof of Arzelà–Ascoli

\[ \Rightarrow \text{ Let } F \text{ be normal.} \]

(1) \(F \) locally bounded

Let \( K \subseteq U \) compact. We show \( F \) bounded, i.e.

\[ \exists M > 0 \forall f \in F \Rightarrow \| f \|_K < M. \]

Assume not for a contradiction. Then

\[ \forall M > 0 \exists f_m \in F \text{ with } \| f_m \|_K \geq M. \]

Letting \( M = n \), we obtain a sequence \( f_n \) with \( \| f_n \|_K \geq n \).

Since \( F \) normal, we can find a subsequence \( f_{n_k} \) converging to \( f \).

Thus \( \| f_{n_k} - f \|_K < \frac{1}{k} \) if \( k \) sufficiently large.

Not \( f_{n_k} \) continuous \( \Rightarrow f \) continuous. so \( \| f \|_K < M \). Then

\[ M > \| f \|_K \geq \| f_{n_k} \|_K - \| f_{n_k} - f \|_K \geq n_k - \frac{1}{k} \rightarrow \infty \text{ as } k \rightarrow \infty \]

This gives a contradiction.
(2) \( F \) locally equicontinuous

Let \( K \subseteq \mathbb{R} \) compact. We show \( \bar{F} \) equicontinuous, that is \( \forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in K \), \( |x - y| < \delta \Rightarrow \forall f \in F \) then

\[
|f(x) - f(y)| < \varepsilon.
\]

Assume not, then

\[
\exists \varepsilon > 0 \exists x, y \in K \text{ with } |x - y| < \delta \forall f \in F \text{ but } |f(x) - f(y)| \geq \varepsilon.
\]

Take \( \delta = \frac{1}{n} \). Then

\[
\exists x_n, y_n \in K, \quad |x_n - y_n| < \frac{1}{n} \quad \forall f_n \in \bar{F} \text{ with } |f_n(x_n) - f_n(y_n)| \geq \varepsilon.
\]

After passing to a subsequence & relabelling, we arrange

\[
\lim_{n \to \infty} f_n \to f \quad \text{because } \bar{F} \text{ normal}
\]

\[
|x_n - y_n| < \frac{1}{n}
\]

\[
|f_n(x_n) - f_n(y_n)| \geq \varepsilon.
\]
Using $f_n$ continuous, $f_n \to f$ we get $f$ continuous.

Since $K$ compact $\Rightarrow f|_K$ uniformly continuous.

Then $\exists \varepsilon > 0$ with

$$|x - y| < \varepsilon, \ x, y \in K \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}. \ (1)$$

Let $N$ be so that $|n - N| \leq N$, we have $\frac{1}{n} < \varepsilon$ and

$$||f_n - f||_K < \frac{\varepsilon}{3}. \ (2)$$

Then $|x_n - y_n| < \frac{1}{n} < \varepsilon \Rightarrow |f(x_n) - f(y_n)| < \frac{\varepsilon}{3}$ by (4).

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{3} \text{ and } |f_n(y_n) - f(y_n)| < \frac{\varepsilon}{3} \text{ by (2).}$$

By triangle inequality (see picture)

$$|f_n(x_n) - f_n(y_n)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This contradicts
The Converse

Assume $F$ is locally equicontinuous & locally bounded.

$\Rightarrow F$ normal

Let $f_n \in F$. We wish to find a subsequence converging locally uniformly?

How do we find such a subsequence?

Plan

- I. arrange pointwise convergence of $f_n$
  
- II. show local uniform convergence

Better Plan

- I. arrange pointwise convergence of $f_n$ only at a countable dense set
  
- II. show local uniform convergence
Let \( \{a_k\} \) be the set of points in \( \mathbb{R} \) with rational coordinates enumerated in any order. Dense!

**Claim** II: After passing to a subsequence of \( f_n \) and relabelling, we may assume

\[ (* ) \quad \forall x \quad \text{the sequence } f_n(a_k) \text{ converges as } n \to \infty. \]

**Claim** III: If \( \{f_n\} \) is equicontinuous & \((*) \Rightarrow f_n \text{ converges locally uniformly.} \)

**We win!**
Proof of Claim II: Cantor diagonalization

We only use pointwise boundedness of \{f_n\}.

Consider \(f_1(a_1), f_2(a_2), \ldots, f_n(a_n), \ldots\) bounded.

Find a subsequence \((s_i)\) \(f_{s_1}, f_{s_2}, \ldots, f_{s_n}, \ldots\) converges at \(a_i\).

Look at the values of \((s_i)\) at \(a_2\) and repeat. We find \(f_{s_2}, f_{s_2}, \ldots, f_{s_n}, \ldots\) converges at \(a_2\) and \(a_i\).

Look at the values of \((s_i)\) at \(a_3\) and repeat.
We obtain an array:

\[
\begin{array}{cccc}
\cdots & f_2 & \cdots & f_n \\
(5_1) & f_{11} & f_{12} & \cdots & f_{1n} \\
(5_2) & f_{21} & f_{22} & \cdots & f_{2n} \\
(5_3) & f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

Each row is a subsequence of the previous one.

Consider the diagonal subsequence

\[
f_{11}, f_{22}, f_{33}, \cdots f_{nn}, \ldots
\]

It is a subsequence of the original sequence. It converges at each \( a_k \). Indeed

\[
f_{11}, f_{22}, \cdots f_{k-1,k-1}, f_{kk}, f_{k+1,k+1}, \cdots
\]

initial terms

part of \( (5_k) \) so we have convergence at \( a_k \).
Proof of Claim 14

Know \( \{a_n\} \) dense in \( U \) and

\[ \forall a, \quad \text{the sequence} \quad \{f_n(a_n)\} \quad \text{converges} \]

(5) \( f_n \) locally equicontinuous

\[ \forall a \in U, \quad \exists \Delta = \text{bounded open ball in } U, \quad a \in \Delta \quad \text{converges uniformly}. \]

(1) \( \forall a \in \Delta, \quad f/\Delta \quad \text{equicontinuous}. \)

Thus \( \forall \varepsilon \exists \delta : \forall 1 \leq \varepsilon/3, \quad x, y \in \delta, \quad f \in \mathcal{F} \)

\[ |f(x) - f(y)| < \varepsilon/3 \]

(2) \( \overline{\Delta} \) can be covered by \( \Delta_i = \Delta(a_i, \delta) \) for \( a_i \in \delta \)

This because \( \{a_i\} \cap \overline{\Delta} \) is dense in \( \overline{\Delta} \).

By compactness, we may assume

\[ \overline{\Delta} \subseteq \bigcup_{i=1}^{e} \Delta(a_i, \delta). \]
(3) Since \( \{ f_n(a_i) \} \) is convergent, it is Cauchy. Hence
\[
\exists N \in \mathbb{N} \quad \forall n, m \geq N \quad \forall 1 \leq i \leq l \quad |f_n(a_i) - f_m(a_i)| < \varepsilon/3
\]

(4) Let \( z \in \overline{\Delta} \). By (2), \( \exists i \) with \( |z - a_i| < \delta \). Let \( n, m \geq N \) as in (5). Then
\[
|f_n(z) - f_m(z)| \leq |f_n(z) - f_n(a_i)| + |f_n(a_i) - f_m(a_i)| + |f_m(a_i) - f_m(z)|
\]
\[
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3
\]

(5) Conclusion
\[
\|f_n - f_m\|_{\overline{\Delta}} < \varepsilon \quad \forall n, m \geq N.
\]

\[
\Rightarrow \{ f_n \} \text{ uniformly Cauchy in } \overline{\Delta}
\]

Lemma
\[
\Rightarrow \{ f_n \} \text{ converges uniformly in } \overline{\Delta}.
\]
This completes the proof.
Remark The converse only used pointwise boundedness.

\[ F \text{ normal} \iff \overline{F} \text{ pointwise bounded + locally equicont} \]

\[ \iff \overline{F} \text{ locally bounded + locally equicont}. \]

The second version bears connections with Montel and it is more uniform.