0. Logistics

(1) Poll regarding Math 220c

MW 3 - 4:20

live / recorded / half live - half recorded?

(2) Midterm - Friday 12 - take home

will cover everything up to and including Monday

Conflicts?

Topics we covered:

- Infinite Products, Γ function, sine
- Weierstrass factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz Lemma & applications
The goal is to frame the discussion & formulate guiding questions.

Given \( u, v \in \mathbb{C} \) we wish to study holomorphic \( f: u \rightarrow v \).

This may be too general. We can ask:

- injective
- finite to one
- bijective
- proper ... etc.
We will focus on bijective holomorphic maps.

**Remark**

1a. Final Exam, Math 220A, we showed

Let $U \subset \mathbb{C}$ be an open set containing 0. Let $f : U \to \mathbb{C}$ be an injective holomorphic function. Show that $f'(0) \neq 0$.

Thus $f : u \to \mathbb{C}$ injective $\implies f'$ has no zeroes.

1b. In Math 220A, Lecture 11, we showed

**Example** $f : u \to \mathbb{C}$ bijective, holomorphic & $f'(u) \neq 0$

for $u \in U$. Then $f^{-1}$ holomorphic.

**Conclusion** $f : u \to \mathbb{C}$ holomorphic & bijective

$\implies f^{-1}$ holomorphic

Bi-holomorphism $=$ holomorphic + bijective
We focus on biholomorphisms.

**Question**

Given $u, v \in \mathbb{C}$ are $u, v$ biholomorphic?

**Remark**

This has implications in topology & differential geometry. In particular $u, v$ are homeomorphic, diffeomorphic.

**Example**

- $u = 0, \ v = \Delta(0,1), \ u \not\sim v$. This follows by Liouville's theorem.

- $u = \mathcal{H}^+, \ v = \Delta, \ c: \mathcal{H}^+ \to \Delta$. Math 220A

Cayley transform: $c(z) = \frac{2 - iz}{z + i}, \ c^*(w) = i \cdot \frac{1 - w}{1 + w}$.

$\mathcal{H}^+ \cong \Delta$.
Very Important Theorem (Riemann Mapping Theorem)

Given $u, v \neq \mathbb{C}$, $u, v$ simply connected $\Rightarrow u, v$ are biholomorphic.

In particular, if $v = \Delta(0,1)$, then any $u \neq \mathbb{C}$ simply connected then $u$ is biholomorphic to $\Delta(0,1)$. 

This is Homework 2, Math 220A.
Riemann’s dissertation (1851) sketched a proof referred by Gauss

"The whole is a solid work of high quality, not merely fulfilling the requirements usually set for doctoral theses, but far surpassing them."

It took the effort of many great minds Wegert, Carathéodory, Hilbert, Schwarz, Koobe, Fejer, Riesz & others.
Question B

Given \( U, V \subseteq \mathbb{C} \) biholomorphic can we construct

- one biholomorphism \( u \rightarrow v \) explicitly?
- all biholomorphism \( u \rightarrow v \) explicitly?

Special cases of (ii)

We saw some specific examples above e.g. the Cayley transform for \( \mathbb{G}^+ \) and \( \Delta(0,1) \).
When \( u = V \), Question B \( \mathbb{W} \) becomes.

**Question C**

What are all biholomorphisms \( f: u \rightarrow u \)?

**Remarks**

\[ \text{Aut}(u) = \{ f: u \rightarrow u : f \text{ holomorphic & bijective} \} \]

is a group. Indeed \( f \in \text{Aut}(u) \Rightarrow f^{-1} \in \text{Aut}(u) \) using that \( f^{-1} \) is automatically holomorphic by the above remarks.

**Examples:** We can consider this question for \( u = \Delta, \mathbb{D}^+, \mathbb{C}, \mathbb{D}^x, \mathbb{C}^x \) etc...
If \( f, g : u \rightarrow v \), \( f \) is known biholomorphism, then any other biholomorphism \( g : u \rightarrow v \) differs from \( f \) by automorphisms:

\[
\Phi \in \text{Aut}(V)
\]

Indeed, \( \Phi = g \circ f ^{-1} \).

In the same fashion

\[
g = f \circ \psi \quad \text{where} \quad \psi = g \circ f ^{-1}
\]

and \( \psi \in \text{Aut}(U) \).

Thus knowledge of Question c helps with aspects of Question B.
**Question D**

Is the action of $\text{Aut}(u)$ on $u$ transitive i.e.

\[ u, b \in U \exists f \in \text{Aut}(u) \text{ with } f(a) = b \?

**Example** $u = \mathbb{C} \setminus \{0\}$. FLT are automorphisms of $u$.

A action is transitive. (Math 220A)

**Question E**

Given $a \in U$, describe $f : U \to U$ biholomorphism, with $f(a) = a$.

Many other questions can be asked.
We begin the discussion with the case

\[ \Omega = \Delta (0,1) = \Delta. \]

The crucial statement is the **Schwarz Lemma**

**Theorem**  Given \( f: \Delta \to \Delta, \Delta = \Delta (0,1) \) holomorphic, \( f(0) = 0 \).

then \( |f'(0)| \leq 1 \) and

\( |f'(z)| \leq |z| \).

If \( |f'(0)| = 1 \) or if \( |f'(z)| = |z| \) for some \( z \in \Delta \setminus \{0\} \) then \( f \) is a rotation, \( f(z) = e^{i\theta} z, \ \theta \in \Delta \).

**Proof** \ - next time.