January 6, 2021
Last time - Infinite products

Conway VII. 5

Given \( p_k \in \mathbb{C} \), define \( P = \prod_{k=1}^{\infty} p_k \) a convergent product if \( \exists N \) with

\[
\lim_{n \to \infty} \prod_{k=N}^{n} p_k = \hat{p} \neq 0 \quad \text{and} \quad \text{set}
\]

\[
P = p_1 \cdots p_{N-1}, \quad \hat{p} = \text{value independent of } N.
\]

Remarks

\[ 1 \] \exists finitely many zero terms

\[ 2 \] \( P = 0 \iff \exists k \) with \( p_k = 0 \)

\[ 3 \] Note

\[
P^n = \frac{\prod_{k=1}^{N} p_k}{\prod_{k=N+1}^{n} p_k} \longrightarrow \frac{\hat{p}}{p} = 1, \quad \text{as} \quad n \to \infty.
\]

Henceforth, we will assume \( p_k = 1 + a_k, \quad a_k \to 0 \).
We seek to connect infinite products to infinite series.

Recall principal branch of logarithm \( z \neq 0, z \in \mathbb{C}\setminus \mathbb{R}_- \)

\[
\begin{align*}
\log (z) &= \log r + i \theta \\
\theta &\in (-\pi, \pi)
\end{align*}
\]

Log \((1+2)\), makes sense if \( z \) small since \( 1+2 \notin \mathbb{R}_- \).

Lemma \:\: \sum_{k=1}^{\infty} (1 + a_k) \text{ converges} \iff \exists \: N > 0 \text{ such that} \sum_{k=N}^{\infty} \log (1 + a_k) \text{ converges} \]

Proof

\[
S_n = \sum_{k=1}^{n} \log (1 + a_k) \implies e^{S_n} = P_n.
\]

\[
P_n = \prod_{k=1}^{n} (1 + a_k)
\]
Proof \(\Leftarrow\) If \(s_n \to s\), \(p_n = \epsilon s_n \to \epsilon s = \hat{P} \neq 0\).

\[\Rightarrow\] Assume \(p_n \to \hat{P}\). We wish to show \(s_n \to s\).

Pick \(\alpha\) such that \(\hat{P} \notin \mathbb{R}_{\leq 0} \exp i\pi\). We use the branch \(\log_\alpha\)

\[\log_\alpha z = \log r + i\theta, \quad \theta \in (\alpha, \alpha + 2\pi)\]

\[e^{s_n} = p_n \Rightarrow s_n = \log_\alpha p_n + 2\pi i l_n, \quad l_n \in \mathbb{Z}.
\]

We claim \(l_n = l_{n-1}\) if \(n > 0 \Rightarrow j \neq l, \quad l_n = l_0\).

\[\Rightarrow s_n = \log_\alpha p_n + 2\pi i l_n \to \log_\alpha \hat{P} + 2\pi i l_0 = j\]

To prove the claim, consider

\[s_n - s_{n-1} = \log_\alpha p_n - \log_\alpha p_{n-1} + 2\pi i (l_n - l_{n-1})\]

\[\downarrow\]

\[0 \quad \downarrow \quad 0 \quad \text{as } n \to \infty\]

Note \(s_n - s_{n-1} = \log (1 + a_n) \to \log 1 = 0\)

\[\log_\alpha p_n - \log_\alpha p_{n-1} \to \log_\alpha \hat{P} - \log_\alpha \hat{P} = 0\]

This shows \(l_n - l_{n-1} \to 0\) as \(n \to \infty\)

\[l_n - l_{n-1} \in 2\]
**Absolute convergence**

**Question.** How do we define absolutely convergent products \( \prod_{k=1}^{n} p_k \)?

**Wrong Answer.** \( \prod_{k=1}^{n} |p_k| \) converges

But then for \( p_k = (-1)^k \), \( \prod_{k=1}^{n} (-1)^k \) converges absolutely

which is absurd.

**Def.** \( \prod_{k=1}^{n} (1 + a_k) \) converges absolutely iff there exists \( N \in \mathbb{N} \) such that

\[ \sum_{k=N}^{\infty} \log(1 + a_k) \] converges absolutely.
Lemma \textbf{TFAE}

\[ \prod_{k=1}^{\infty} (1 + a_k) \text{ converges absolutely} \]

\[ \sum_{k=1}^{\infty} a_k \text{ converges absolutely} \]

\[ \prod_{k=1}^{\infty} (1 + |a_k|) \text{ converges} \]

**Proof** Consider Taylor expansion in \( D(a, s) \in \mathbb{C}\setminus\{-1, 1\} \)

\[ \log(1 + 2) = 2 - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} + \ldots \]

\[ \frac{\log(1 + 2)}{2} = 1 - \frac{2^2}{2} - \frac{2^3}{3} - \ldots \]

\[ \Rightarrow \lim_{z \to 0} \frac{\log(1 + z)}{2} = 1 \Rightarrow \exists \rho > 0 \text{ such that } \|z\| < \rho, z \neq 0. \]

\[ \frac{1}{2} \leq \left| \frac{\log(1 + z)}{2} \right| \leq \frac{3}{2} \]

**Important inequality** \( \exists \rho \text{ s.t. if } \|z\| < \rho \)

\[ \frac{1}{2} \|z\| \leq \left| \log(1 + z) \right| \leq \frac{3}{2} \|z\|. \]
By defn, \( \prod_{k=1}^{\infty} (1+q_k) \) converges absolutely

\[ \iff \sum_{k=n}^{\infty} \log (1+q_k) \text{ converges absolutely} \]

\[ \iff \sum_{k=n}^{\infty} a_k \text{ converges absolutely} \ (\text{comparison test} + \text{important inequality}) \]

Finally,

\[ \iff \sum_{k=n}^{\infty} |a_k| \text{ converges absolutely} \]

\[ \iff \prod_{k=1}^{\infty} (1 + |a_k|) \text{ converges absolutely} \text{ by } [\quad] \iff [\quad] \text{ for } \tilde{a}_k = |a_k| \]

\[ \iff \prod_{k=1}^{\infty} (1 + |a_k|) \text{ converges} \iff [\quad] \]

Indeed, absolute convergence of the product is superfluous

\[ \sum_{k=n}^{\infty} |\log (1+q_k)| = \sum_{k=n}^{\infty} \log (1+q_k) \]
Remark (Rearrangements).

Math 140A we learned that if \( \sum_{k=1}^{\infty} b_k \) is absolutely convergent then \( f : \mathbb{N} \rightarrow \mathbb{N} \) bijection

then \( \sum_{k=1}^{\infty} b_{f(k)} \) converges to the same sum.

The same happens for absolutely convergent products

\( \prod_{k=1}^{\infty} f_k \) can be rearranged, \( b_k = \log(1 + a_k) \), \( f_k = 1 + a_k \).
2. Infinite Products of Holomorphic Functions

\( f_k : \mathbb{U} \to \mathbb{C}, \quad \mathbb{U} \subseteq \mathbb{C} \)

**Assumption**

\[ \sum_{k=1}^{\infty} |f_k| \text{ converges locally uniformly} \]

**Terminology**

\[ \sum_{k=1}^{\infty} f_k \text{ converges absolutely locally uniformly} \]

**Define**

\[
(4) \quad F(z) = \prod_{k=1}^{\infty} \frac{1}{1 + f_k(z)}.
\]

**Remark**

(\( \star \)) \( \sum_{k=1}^{\infty} \) converges absolutely \( \forall z \in \mathbb{U} \Rightarrow \text{can rearrange the product.} \)
Proposition: Under the above Assumption

1. the partial products of (*) converge locally uniformly.

2. \( F \) is holomorphic

3. \( F(\infty) = 0 \iff \exists k \text{ with } 1 + f_k(\infty) = 0 \)

Proof will be given next time.

Examples:

1. \[ \sum_{k=1}^{\infty} \frac{1}{11} \left(1 - \frac{2}{k^2}\right) \] defines an entire function with zeroes only at the integers and nowhere else.

   Indeed, apply the Proposition to \( f_k(z) = \frac{2}{k^2} \).

2. \[ \sum_{k=1}^{\infty} \frac{1}{11} \left(1 + 2^{\frac{k}{z}}\right) \] is an entire function if \( |z| < 2 \)

   with zeroes only at \( z = -\frac{1}{2} \).

   Apply the Proposition to \( f_k(z) = 2^{\frac{k}{z}} \).