Extra hints added to 1 ii. & 1 iv.

Office Hour: 4 – 5:30 today.

1. More examples of biholomorphisms

Example 1: Squaring in $\mathbb{H}^+$ "Half $\mathbb{H}^+ \rightarrow \mathbb{H}^+"

$z \rightarrow z^2$. 

$c(z) = \frac{z - i}{z + i}$

c : $\mathbb{H}^+ \rightarrow \Delta$. 

$$c$$
Example 11.1 \[ \Delta^+ = \text{upper half disc (open)} \]

**Question** Find \( \Delta^+ \xrightarrow{\omega} \Delta, \quad 2 \rightarrow 2^2 \)

**Answer** Not done by squaring since \( 0 \in \Delta, \omega \notin \Delta^+ \).

**Instead** We use the Cayley transform & work in \( \mathbb{H}^+ \).

**Idea**

\[ \Delta \xrightarrow{c^{-1}} \mathbb{H}^+ \]

the map we want half disc

\[ \Delta^+ \xrightarrow{c^{-1}} \text{"half" } \mathbb{H}^+ \]

squar
Concretely consider \( c : \mathbb{H}^+ \rightarrow \Delta \), the Cayley transform

\[
c^{-1} : \Delta \rightarrow \mathbb{H}^+,
\]

\[
c^{-1} (i) = i \frac{1 + i}{1 - i}.
\]

Check under \( c^{-1} \), we map

\[
\begin{align*}
-1 & \rightarrow 0 \\
1 & \rightarrow \infty \\
0 & \rightarrow i
\end{align*}
\]

\[
\begin{align*}
i & \rightarrow -1 \\
-1 & \rightarrow +1
\end{align*}
\]
Conclusions

1. Diameter $b \rightarrow$ Imaginary axis $\tilde{b}$

2. $\text{arc } a \rightarrow$ Negative real axis $\tilde{a}$

3. $\text{arc } c \rightarrow$ Positive real axis $\tilde{c}$

4. $\Delta^+ \rightarrow L = 2^{nd}$ quadrant (left)

5. $\Delta^- \rightarrow R = 1^{st}$ quadrant (right)
Construction of biholomorphism $\Delta^+ \to \Delta$ as a composition of three moves:

1. \[ \Delta^+ \xrightarrow{c^{-1}} L, \quad z \to \frac{1+z}{1-z} \]

2. \[ L \xrightarrow{g^*} \Delta^*, \quad z \to -z^2 \]

3. \[ \Delta^* \xrightarrow{c} \Delta, \quad c(z) = \frac{3-z}{3+z} \]

\begin{align*}
\Delta^+ & \quad \xrightarrow{c^{-1}} \\
L & \quad \xrightarrow{g^*} \\
\Delta & \quad \xrightarrow{c} \\
\end{align*}
Conclusion

The biholomorphism $\Delta^+ \rightarrow \Delta$ extends to $\Delta^+ \rightarrow \Delta$ continuously and bijectively.

(the upper arc $a$ is sent to the upper arc & the diameter $b$ is sent to the lower arc).
2. **Extension to the boundary**

**Question**
Given \( f: U \to \Delta \) a biholomorphism, does it extend \( \tilde{f}: \tilde{U} \to \tilde{\Delta} \) bicontinuously?

**Answer**
Yes if \( U \) bounded and \( \partial U \) simple closed curve.

*Caraoteodory’s theorem*

*We will not give the proof in this course.*
3. **Beyond the boundary**

**Question** Can we extend beyond the boundary?

The easiest instance is provided by

Schwarz Reflection Principle

Conway IX. 1.

There are several versions but two stand out:

1. reflection across line segments *(book)*

2. reflection across circular arcs *(HWKC)*

**Applications**

1. biholomorphic maps between rectangles, annuli,

2. analytic continuation...
4. Reflect across segments

Open \( u \subseteq \mathbb{C} \) symmetric \( \overline{z} \mapsto \overline{z} \). \( u \ni \alpha \rightarrow \overline{\alpha} \in u. \)

\[
\begin{align*}
  u^+ &= u \cap f^+ \\
  u^- &= u \cap f^- \\
  u^o &= u \cap R_1 = (a, b)
\end{align*}
\]

\( \mathbb{R} \)

\( u^o \)

\( u^- \)

Given \( f : u^+ \rightarrow \mathbb{C} \)

\( \text{holomorphic in } u^+ \)

extends continuously to \( u^o \).

such that the values \( f(u^o) \subseteq \mathbb{R} \).

Define \( F : \mathbb{C} \rightarrow \mathbb{C} \) by

\[
F(z) = \begin{cases} 
  f(z) & \text{if } z \in u^+ \\
  f(z) & \text{if } z \in u^o \\
  f(\overline{z}) & \text{if } z \in u^-
\end{cases}
\]
**Theorem**

The function $F: U \rightarrow \mathbb{C}$ is a holomorphic extension of $f$ beyond the boundary.

**Remarks**

**Visualization**
The condition \[ f(u_0) \leq \mathbb{R} \]

ensures we reflect across real axis on both sides.

More generally, we can reflect across arbitrary lines.

This can be deduced via rotations.
Using the Cayley transform

\[ C : \Delta \rightarrow \mathbb{H}^+ \]

we can also reflect across arcs in the unit disc.

(HWK 1).