Part I: Weierstrass & Mittag-Leffler

Series & Products

Part II: Riemann & Schwarz

Mapping Theory

Part III: Runge < Conway VIII. 1

Approximation Theory
§1. Context for Runge

In real analysis (Math 140B), we learn the

Weierstrass Approximation Theorem

\[ f : [a,b] \rightarrow \mathbb{R} \text{ continuous, } \exists P_n \text{ polynomials} \]

\[ P_n \rightarrow f. \]

This was proven by Weierstrass at age 70 in 1885.

There are many applications of this theorem.

\( \text{e.g. in Fourier analysis, functional analysis etc.} \)

Remark This can be generalized in \( \mathbb{R}^n \).

If \( K \subseteq \mathbb{R}^n \) compact, \( f : K \rightarrow \mathbb{R} \text{ continuous}, \) then

\[ \exists P_n \text{ polynomials, } P_n \rightarrow f \text{ in } K. \]
Runge (age 28, Ph.D. 1850, student of Weierstrass):

**Question**  
What about $f$ holomorphic? Can it be approximated by polynomials in $z$?

**Answer**  
was given in 1855 as well.

**Remark**  
This doesn’t follow from Weierstrass.

Weierstrass produces polynomials in $x, y$ for $z = x + iy$.

E.g. polynomials in $z$ and $ar{z}$.
Carl Runge (1856 - 1927)

- Runge - Kutta
- Runge's Approximation
- mathematics, astrophysics, spectroscopy.
2. Phrasing the Question more carefully

Beware A holomorphic function is defined over open sets. (see Math 220A).

Definition \( K \subseteq \mathbb{C} \) compact. A holomorphic function in \( K \) is a function \( f: K \rightarrow \mathbb{C} \) that extends holomorphically to a neighborhood \( U \ni K \).
Two versions of the question

**Runge C** (compact sets) \( K \subseteq \mathbb{C} \) compact

Given \( f \) holomorphic in \( K \), are there polynomials \( P_n \) such that \( P_n \to f \) in \( K \)?

**Runge O** (open sets) \( U \subseteq \mathbb{C} \) open

Given \( f \) holomorphic in \( U \), are there polynomials \( P_n \) such that \( P_n \to f \) in \( U \)?
**Emphasis**

**Runge C**: approximation on a single compact $K$

**Runge O**: approximation on all compacts $K$ in the domain of a holomorphic function

**Runge C** is more basic.

Complex analysis

$\xrightarrow{z}$

Runge C $\quad \Rightarrow \quad$ Runge O

Point-set topology.

The two versions are very similar.
Example Runge C.

\[ K = \{ 1 \leq |z| \leq 2 \} \text{, } f(z) = \frac{1}{z} \text{ holomorphic in } K. \]

Can we find \( P_n \rightarrow f \) in \( K \)?

No! Note \( f \) is holomorphic in \( u \supset K, \quad u = \left\{ \frac{1}{2} < |z| < \frac{5}{2} \right\} \)

so "holomorphic in \( K \)."

If \( P_n \rightarrow f \) in \( K \) then

\[ \int_{|z|=\frac{3}{2}} P_n \, dz \rightarrow \int_{|z|=\frac{3}{2}} f \, dz. \]

Note \( \int_{|z|=\frac{3}{2}} P_n \, dz = 0 \) \& \( \int_{|z|=\frac{3}{2}} f \, dz = 2\pi i \) by the residue theorem. This is a contradiction.

The failure is due to the "hole" in \( K \).
What is a "hole"?

**Definition** \( K \subseteq \mathbb{C}, \text{ compact} \)

A hole is a bounded connected component of \( \mathbb{C} \setminus K \).

**Example**

\[ K \quad \mathbb{C} - K = A \cup B \cup C \]

\( C \) unbounded

\( A, B \) bounded

\( A, B \) are holes for \( K \).

\[ K = \bigcup \left\{ \{1/n \mid n \geq 1\} \cup \{0\} \right\} \]

\( K \) closed & bounded \( \Rightarrow \)

\( \Rightarrow K \) compact.

\( \infty \) - many holes

\[ H_n = \left\{ \left. \frac{1}{n+1} < |z| < \frac{1}{n} \right\} \right. \]
3. **Runge's Theorem - Compact Sets**

We give three versions. The simplest version is:

**Runge's Little Theorem (Case c)**

If $K$ has no holes ($\iff K \setminus c$ connected), then for every holomorphic function $f$ in $K$, there exist polynomials $P_n$ with

$$P_n \rightarrow f \text{ in } K.$$
**Question:** How about arbitrary $K$?

**Answer:** Polynomial approximation fails (Example)

Are we even asking the right question?

**Better**

Rational Approximation.

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**Question:** Given $f$ holomorphic in $K$,

\[ \exists \text{ rational functions, } R_n \rightarrow f \text{ in } K \& \]

pole of $R_n$ are outside $K$?

**Question:** Can we prescribe the location of the poles of $R_n$?
Runge's (Almost final) $K \subseteq \mathbb{C}$ compact.

**Theorem** Let $S$ be a set of points, at least one from each hole of $K$.

Then $\exists f$ holomorphic in $K$.

**Claim** $\exists R_n \rightarrow f$ in $K$.

**Remark** $R_n$ are rational functions whose poles are in $S$.

**Remark** The poles of $R_n$ are contained in $S$, but it may happen that not all points of $S$ are poles.

**Remark** If $K$ has no holes then $S = \mathbb{C}$. Thus $R_n$ has no poles $\Rightarrow R_n$ have no denominators $\Rightarrow R_n$ are polynomials. We recover Little Runge.
We replace $C$ by $\hat{C} = C \cup \{\infty\}$.

Theorem: Let $K \subseteq \mathbb{C}$, compact. Let $S \subseteq \hat{C}$ be a set of points, at least one chosen from each component of $\hat{C} \setminus K$. Let $f$ be holomorphic in $K$. Then

$$\exists R_n \rightarrow f \text{ in } K$$

in $\hat{C}$

$R_n$ are rational with possible poles in $S$. 
Remark: An interesting case allowed by the Final Version is to pick \( v \in S \) from the unbounded component. Thus, when \( S \) consists in

- \( \infty \) from the unbounded component of \( \mathbb{C} \setminus K \)
- a point from each bounded component of \( \mathbb{C} \setminus K \) (holes)

we recover the Almost Final Version.

The two versions are even equivalent in this case since the condition that a rational function \( R \) have at worst a pole at \( \infty \) is vacuous. Indeed,

\[
R(z) = \frac{\prod_{i=1}^{m} (z-a_i)}{\prod_{i=1}^{n} (z-b_i)} \Rightarrow R\left(\frac{1}{2}\right) = 2^{m-n} \frac{\prod_{i=1}^{m} (1-a_i \cdot 2)}{\prod_{i=1}^{n} (1-b_i \cdot 2)}
\]

has at worst a pole at 0.
Summary

Runge C (Final) \rightarrow Runge C (Almost Final)

Conway VIII.1.7

\[ \square \]

- rational approximation
- version for \( \mathbb{C} \)

\[ \square \]

- rational approximation
- poles in each hole

\[ \square \]

Little Runge C

- polynomial approximation
- \( K \) has no holes
Example Review

\[ f(z) = \frac{z^3}{(z-2)(z-7)} \]

\[ K = \left\{ 3 \leq |z| \leq 4 \right\} \]

\[ f \text{ is holomorphic in } K \text{ because it extends holomorphically to } \]

\[ U = \left\{ \frac{5}{2} < |z| < \frac{9}{2} \right\} \supseteq K. \]

Can we approximate \( f \) uniformly on \( K \) by:

1. rational functions with poles at 1?

**Yes** Almost Final Version

2. rational functions with poles at 0, \( \infty \)

**Yes** Final Version

3. rational functions with poles at \( \infty \)?

**No.** Such rational functions would have to be
polynomials (if they had denominators, there would be poles). But if \( P_n \rightarrow f \) then

\[
\int P_n \, d\alpha \rightarrow \int f \, d\alpha = 2\pi i \cdot \text{Res}(f, 2)
\]

\[
|z| = \frac{\pi}{2} \quad |z| = \frac{\pi}{2}
\]

0

\[
= 2\pi i \cdot \left| \frac{2^3}{z^2 - 7} \right| \quad z = 2
\]

using the Residue theorem. Contradiction!