Last time:

\[ f_k : u \rightarrow \mathbb{C} \text{ holomorphic} \]

\[ \sum_{k=1}^{\infty} |f_k| \text{ converges locally uniformly} \]

(1) \[ h(z) = \frac{1}{\prod_{k=1}^{\infty} (1 + f_k(z))} \text{ holomorphic} \]

(2) \[ \frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{1 + f_k} \]

The series on RHS converges absolutely locally uniformly on \( u \setminus \text{Zero}(h) \).

Remark:

If \( \sum_{k=1}^{\infty} |\text{Log}(1 + f_k)| \text{ converges locally uniformly} \)

the same conclusions hold.
Today [1] factorization of sinc

Euler, 1734

These two topics are naturally connected
Factorization of sine \( \text{ (Euler, 1734) } \)

**Theorem**

\[
\sin \pi x = \pi x \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{k^2} \right)
\]

**Idea:** Both sides have the same zeroes (with multiplicity)

**Question** When do two entire functions have exactly the same zeroes?

**Lemma** If \( f, g : \mathbb{C} \to \mathbb{C} \) entire, with the same zeroes and multiplicities, then \( f = g e^h \) for some \( h : \mathbb{C} \to \mathbb{C} \) entire.
Proof. Let \( H = \frac{f}{g} \). \( H \) entire without zeroes by hypothesis. We show \( H = e^h \).

The function \( \frac{H'}{H} \) is entire so it admits primitive \( k \).

\[ \Rightarrow \frac{H'}{H} = h' \text{. Then} \]

\[ (He^{-h})' = H' e^{-h} - H e^{-h} h' = e^{-h}(H' - H h') = 0 \]

\[ \Rightarrow H e^{-h} = c \neq 0 \Rightarrow H = c e^h = e^{\log c + h} \]

Remark. The same holds for \( f, g \) as \( u \to c \), \( u \) simply connected.
Proof of the sine factorization

(1) Convergence:

Note that \[ \sum_{k=1}^{\infty} \frac{2^2}{2^2} \] converges locally uniformly \( \Rightarrow \sum_{k=1}^{\infty} \frac{1}{1} \left( 1 - \frac{2^2}{2^2} \right) \) converges.

(2) Location of zeroes:

Both sides \( \sin \pi x \) & \( \prod_{k=1}^{\infty} \left( 1 - \frac{2^2}{2^2} \right) \) have simple zeroes at the integers & nowhere else.

(3) Completing the proof:

By the Lemma, \( f(x) \) entire

\[ \sin \pi x = \pi x \prod_{k=1}^{\infty} \left( 1 - \frac{2^2}{2^2} \right) \]

We show \( \beta = 0 \). Compute logarithmic derivative

\[ \frac{\pi \cos \pi x}{\sin \pi x} = \frac{\left( e^\beta \right)'}{e^\beta} + \frac{\pi}{\pi x} + \sum_{k=1}^{\infty} \frac{-2^2}{1 - \frac{2^2}{2^2}} \]

\[ \pi \cot \pi x = \pi' + \frac{1}{\pi x} + \sum_{k=1}^{\infty} \frac{2^2}{x^2 - \frac{2^2}{2^2}} \]
Recall Math 220, HWK 6:

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let $\gamma_n$ be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$
\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2} \, dz
$$

via the residue theorem. Making $n \to \infty$, show that

$$
\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.
$$

Thus $h' \equiv 0 \Rightarrow h$ constant. We show $h \equiv 0$.

From

$$
\frac{\sin \pi z}{\pi z} = e^{\frac{1}{z}} \prod_{k=1}^{\infty} \left(1 - \frac{2^2}{k^2}\right), \quad \text{make } z \to 0
$$

$$
1 = e^{h(0)} \cdot 1 \Rightarrow h(0) = 0 \Rightarrow h \equiv 0.
$$

This completes the proof.
**Remark**  
\[ \sin \pi \frac{2}{2} = \frac{\pi}{2} \left( 1 - \frac{2^2}{k^2} \right) \]

[1] \[ \frac{\pi}{2} = \frac{1}{2} \]

\[ 1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left( 1 - \frac{1}{4k^2} \right) = \frac{\pi}{2} \prod_{k=1}^{\infty} \frac{(2k^2)}{(2k-2)(2k+2)} \]

\[ \Rightarrow \frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k^2)}{(2k)(2k+1)} \]

\[ \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \ldots \]

Wallis, 1655

[11] \[ \frac{\pi}{2} = \pi \]

\[ \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k^2} \right) = \frac{\sin \pi \frac{2}{2}}{\pi \frac{2}{2}} = e^{\pi} - e^{-\pi} \]

[111] \[ \cos \pi \frac{2}{2} = \frac{\sin \pi \frac{2}{2}}{2 \sin \pi \frac{2}{2}} = \frac{2 \pi}{2} \prod_{k=1}^{\infty} \left( 1 - \frac{4 \frac{2^2}{k^2}}{2 \pi^2} \right) \]

Splitting into even and odd:

\[ \cos \pi \frac{2}{2} = \prod_{k=1}^{\infty} \left( 1 - \frac{4 \frac{2^2}{k^2}}{(2k-1)^2} \right) \]
2. **Γ**: function — probability, statistics, combinatorics, ...

"The product 1.2. ... \( x \) is the function that must be introduced in analysis." (Gauss to Bessel, 1811)

\[
\prod_{n=1}^{x} n = "1 \cdot 2 \cdot 3 \ldots \cdot x" = \Gamma(x + 1)
\]

"The theory of analytic factorials does not seem to have the importance some mathematicians used to attribute to it".

Worsthoff 1854

**Definition**

\[
G(2) = \frac{1}{\Pi} \left( 1 + \frac{2}{n} \right)^n e^{-2/n}
\]

**Remark**

The convergence (absolutely & locally uniformly) of the product is HWK 1, #4. There, you show

\[
\sum_{n=1}^{\infty} \left| \log \left( 1 + \frac{2}{n} \right) e^{-2/n} \right| \text{ converges locally uniformly.}
\]
Properties of the function $G$

**Theorem:**

\[
G(z) \cdot G(-z) = \prod_{n=1}^{\infty} \left(1 + \frac{2}{n^2}\right) = e^{\gamma^2} \cdot \frac{1}{\zeta(2)}
\]

where $\gamma$ is Euler constant.

\[
\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right).
\]

We will prove this next time.

**Definition:**

\[
\Gamma(z) = \frac{e^{\gamma^2} \cdot \frac{1}{\zeta(2)}}{2 \cdot \pi^2 \cdot G(z)}
\]

**Remark:**

$G$ has zeroes at $-1, -2, \ldots, -n, \ldots$

$\Rightarrow \Gamma$ meromorphic in $\mathbb{C}$ with zeroes at $-1, -2, \ldots, -n, \ldots$