Math 220B — Lecture 8

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Weierstrass Problem for arbitrary regions

Question Given $u \in \mathbb{C}$, $\{a_n\} \subseteq u$ without limit point in $u$, find $f$ holomorphic in $u$ with zeroes only at $\{a_n\}$.

The sequence $\{a_n\}$ may contain repetitions according to multiplicities of the zeroes.

Main Theorem The Weierstrass Problem can be solved in $u$. 
Remark II. It is not true any two solutions $f_1, f_2$ satisfy

$$f_1 = e^{h} f_2$$

Counterexample

$U = \mathbb{C}^*$, $f_1 = 1$, $f_2 = 2$.

$h$ would have to be a logarithm, which is undefined in $\mathbb{C}^*$.

Any meromorphic function in $U$ is quotient of two holomorphic functions.

The same proof for $U = \mathbb{C}$ works for all $U$. 
How to prove Weierstrass for $u$?

We could again try

$$f(x) = \prod_{n=1}^{\infty} p_n \left( \frac{x}{a_n} \right).$$

Convergence used $a_n \to \infty$.

Indeed, if we wish to have

$$\sum_{n=1}^{\infty} \left( \frac{x}{la_n} \right)^{p_n+1} < \infty \quad \text{we’d need} \quad \frac{x}{la_n} \to 0 \Rightarrow a_n \to \infty.$$

Since $a_n \in u$, this may not be the case, e.g. if $u$ is bounded. How to deal with bounded regions for instance?
New ideas

1) Use biholomorphisms to change the region \( U \).

\[ \text{e.g. via } z \rightarrow \frac{1}{z}. \]

If \( U \) were bounded, the new region would be unbounded.

2) Think of \( U \subseteq \mathbb{C} \) as \( \tilde{U} \subseteq \hat{\mathbb{C}} \) and prescribe values at \( \tilde{\infty} \) as well.

New idea

Even for unbounded regions, we can try new functions:

\[ f(z) = \prod_{n=1}^{\infty} E_n \left( \frac{a_n - b_n}{z - b_n} \right) \]

for good choices of \( b_n \).

This also has zeroes at \( z = a_n \) since \( E_n(1) = 0 \).
Weierstrass Problem in \( \mathbb{C} \)

**Step (1)** Assume \( \exists R > 0 \) in neighborhood of \( \infty \).

\[
\{ |z| \geq R \} \subset \mathbb{U}
\]

\[
|a_n| \leq R \quad \forall n.
\]

Construct \( f \) holomorphic in \( \mathbb{U} \) such that

\[
f \text{ has zeroes at } a_n
\]

\[
\lim_{z \to 0} f(z) = L.
\]

**Step (2)** General case. Use easy trick.

Use \( z \to \frac{1}{z} \) to reduce to Step 1.
Topological Fact used in the Proof (Rudin)

\[ K \cap F = \emptyset, \ K \neq \emptyset \text{ compact, } F \neq \emptyset \text{ closed.} \]

\[ d = \text{dist}(K, F) = \inf_{k \in K} \inf_{f \in F} |k - f| > 0 \]

**Proof**

Assume \( d = 0 \). Then \( \exists k_n \in K, f_n \in F \) with

\[ |k_n - f_n| \to 0 \]

Passing to a subsequence, assume \( k_n \to k \in K \).

It follows that \( f_n \to k \) as well.

Since \( F \) closed, \( k \in F \). Thus \( k \in K \cap F = \emptyset \). Contradiction.
\textbf{Step 1:} \( \exists R > 0, \{ \lfloor R \rfloor \} \subseteq \mathbb{Z} \) & \( |a_n| \leq R \).

\( \text{Claim:} \) \( f \) zero at \( a_n \) only.

\( \lim_{z \to a} f(z) = 1 \)

Note \( K = \mathbb{C} \setminus \mathbb{U} \subseteq \{ |z| \leq R \} \Rightarrow K \) bounded & closed

\( \Rightarrow K \) compact.

Since \( |a_n - 2| \) is continuous, \( \exists b_n \in K \) with

\[ |a_n - b_n| = \min_{z \in K} |a_n - z| . \]

Write \( S_n = |a_n - b_n| > 0 \) since \( a_n \in \mathbb{U} \), \( b_n \notin \mathbb{U} \).

\( \text{Claim:} \) \( S_n \to 0 \).

\textbf{Proof:} Assume otherwise. Then \( \exists \varepsilon > 0 \) \( \exists N \geq N \) with

\[ |S_n| \geq \varepsilon . \]

Passing to a subsequence we may assume \( |S_n| \geq \varepsilon \) for.
Note $\{a_n\} \subseteq \overline{B}(0, R) = \text{compact}$. Passing to a subsequence we may assume $a_n \to a$. Since $\{a_n\}$ has no limit point in $U \implies a \in K$. Then by the definition of $b_n$:

$$|a_n - a| \geq |a_n - b_n| = \delta_n > \varepsilon.$$  

This contradicts $a_n \to a$. Thus $S_n \to 0$. 
Claim: \( f(z) = \prod_{n=1}^{\infty} E_n \left( \frac{a_n - b_n}{z - b_n} \right) \) converges absolutely and locally uniformly in \( U \) & vanishes only at \( a_n \).

Proof: It suffices to show

\[
\sum_{n=1}^{\infty} \left| E_n \left( \frac{a_n - b_n}{z - b_n} \right) - 1 \right| \text{ converges absolutely and locally uniformly in } U.
\]

To this end, let \( k' \subseteq U \) compact.

Let \( \delta = d(K, K') > 0 \) since \( K \cap K' = \emptyset \).

Let \( z \in K' \Rightarrow |z - b_n| \geq \delta \Rightarrow \)

\[
\left| \frac{a_n - b_n}{z - b_n} \right| \leq \frac{\delta}{\delta} \leq \frac{1}{\delta} \text{ if } n \not\in N \text{ since } \delta_n \to 0.
\]

Recall

\[
|1 - E_p(\omega)| \leq |\omega|^{r+1} + |\omega| \leq 1.
\]

Thus

\[
|1 - E_n \left( \frac{a_n - b_n}{z - b_n} \right)| \leq |a_n - b_n| \left( \frac{1}{\delta^n} \right)^{r+1} \forall \in K', n \geq N
\]

We conclude by Weierstrass M-test since \( \sum_{n}^{1} \frac{1}{\delta^n} < \infty \).
Proof of \[ \lim_{x \to 0} f(x) = 1. \]

Equivocally  \[ \lim_{x \to 0} f\left(\frac{1}{x}\right) = 1. \]

We compute  \[ g(\zeta) = f\left(\frac{1}{\zeta}\right) = \prod_{n=1}^{\infty} E_n \left( \frac{a_n - b_n}{\sqrt{\zeta} - b_n} \right) = \prod_{n=1}^{\infty} E_n \left( \frac{2(a_n - b_n)}{1 - 2b_n} \right). \]  

We show the product \((x)\) converges absolutely and locally uniformly in \(\Delta\left(0, \frac{1}{R}\right)\). The limit will be holomorphic at \(\zeta = 0\) hence continuous. Then

\[ \lim_{\zeta \to 0} g(\zeta) = g(0) = 1 \Rightarrow \lim_{\frac{1}{x} \to 0} f\left(\frac{1}{x}\right) = 1. \]
To show convergence, let \( \Delta (0, \rho) \subseteq \Delta (0, \frac{1}{R}) \). 

We have for \( z \in \Delta (0, \rho) \):

\[
\left| \frac{z (a_n - b_n)}{1 - 2b_n} \right| \leq \frac{\rho \delta_n}{1 - 2b_n} \leq \frac{\rho \delta_n}{1 - 21b_n} \leq \frac{\rho \delta_n}{1 - \rho R} \leq \frac{1}{2}. 
\]

for \( n \geq N \) since \( \delta_n \to 0 \).

Then

\[
\left| 1 - E_n \left( \frac{2 (a_n - b_n)}{1 - 2b_n} \right) \right| \leq \left| \frac{2 (a_n - b_n)}{1 - 2b_n} \right| \to \frac{1}{2} \quad \Rightarrow 
\]

\[ \Rightarrow \text{Weierstrass M-test} \]

\[
\sum_{n=1}^{\infty} \left| 1 - E_n \left( \frac{2 (a_n - b_n)}{1 - 2b_n} \right) \right| \text{ converges absolutely and locally uniformly in } \Delta (0, \frac{1}{R}).
\]
Case (2) General case

WLOG \( o \in U \) \& \( a_n \neq 0 \)

Indeed we may take \( a \in U \), \( a \neq a_n \) \forall n. Let

\[ U^{\text{new}} = \{ u - a, u \in U \}, \quad a_n^{\text{new}} = a_n - a, \]

\[ \Rightarrow 0 \in U^{\text{new}}, \quad a_n^{\text{new}} \neq 0. \]

If \( f^{\text{new}} \) solves \( \text{Weierstrass} \) for \( (U^{\text{new}}, \{a_n^{\text{new}}\}) \) let \( f(x) = f^{\text{new}}(x - a) \), solves \( \text{Weierstrass} \) for \( (U, \{a_n\}) \).

Trick to reduce to Case 1

Define \( \tilde{U} = \{ \frac{1}{x}, x \in U \setminus \{0\} \} \). This is open by the open mapping theorem for \( U \setminus \{0\}, \ x \mapsto \frac{1}{x} \).
Let $\tilde{a}_n = \frac{1}{a_n} \in \tilde{\omega}$.

Claim $(\tilde{\omega}, \{\tilde{a}_n\})$ satisfies Step 1.

Let $f$ be the solution to Weiersratch for $(\tilde{\omega}, \{\tilde{a}_n\})$.

Let \( f(1) = \tilde{f} \left( \frac{1}{2} \right) \) be holomorphic in $u \setminus \{o\}$.

Since $\lim_{2 \to 0} \tilde{f}(2) = 1 \Rightarrow \lim_{x \to 0} f(x) = 1$. Thus $o$ is removable singularity and $f$ extends to $u$. Its zeroes are only at $a_n$.

Proof of the claim

Since $0 \in \omega \Rightarrow \exists \varepsilon \text{ with } \Delta (0, \varepsilon) \subseteq \omega$.

$\Rightarrow \{ (z) \geq \frac{1}{\varepsilon} \} \subseteq \tilde{\omega}$.

Since $0 \in \omega \Rightarrow \{a_n\} \text{ do not have } 0 \text{ as limit point}$

$\Rightarrow \exists \varepsilon' \text{ with } |a_n| \geq \varepsilon' \Rightarrow |\tilde{a}_n| \leq \frac{1}{\varepsilon'}$.

Let \( R = \max \left( \frac{1}{2}, \frac{1}{\varepsilon'} \right) \Rightarrow |a_n| \leq R \& \{ 1 \leq R \} \subseteq \omega$. 


Exercise

Follow the above proof for $u = e$. What function $f$ does the proof produce?