Q1.

(i) Recall (Lecture 15) that any automorphism $f$ of the unit disc is a composition of a rotation with a fractional linear transformation

$$
\phi_a(z) = \frac{z - a}{1 - \overline{a}z},
$$

where $a \in \Delta := \Delta(0, 1)$.

Rotation : Observe that

$$
d(e^{i\theta}z, e^{i\theta}w) = \left| \frac{e^{i\theta}(z-w)}{1 - (e^{i\theta})\overline{e^{i\theta}}w} \right| = d(z, w)
$$

since $e^{i\theta}$ has absolute value 1 and its conjugate is $e^{-i\theta}$.

Fractional linear transformation : Note that $|a| < 1$ and thus $(1 - \overline{a}z)$ and $(1 - \overline{a}w)$ are non-zero quantities for $z, w \in \Delta$. We follow the following calculation to get the result.

$$
d(\phi_a(z), \phi_a(w)) = \left| \frac{\phi(z) - \phi(w)}{1 - \phi(z)\phi(w)} \right|
= \left| \frac{(z - a)(1 - \overline{a}w) - (w - a)(1 - \overline{a}z)}{(1 - a\overline{z})(1 - \overline{a}w) - (\overline{z} - a)(w - a)} \right|
= \left| \frac{(z - w)(1 - |a|^2)}{(1 - \overline{z}w)(1 - |a|^2)} \right|
= d(z, w).
$$

(ii) We will reduce the problem to the Schwarz lemma. Fix $w \in \Delta$. Let

$$
g = \phi_{f(w)} \circ f \circ \phi_w^{-1}
$$

and note

$$
g(0) = \phi_{f(w)} \circ f \circ \phi_w^{-1}(0) = \phi_{f(w)} \circ f(0) = 0.
$$

By Schwarz Lemma

$$
d(g(z), 0) \leq d(z, 0)
$$

using that $d(u, 0) = |u|$. Then

$$
d(f(z), f(w)) = d(\phi_{f(w)} \circ f(z), \phi_{f(w)} \circ f(w))
= d(g \circ \phi_{w}(z), 0)
\leq d(\phi_{w}(z), 0)
= d(\phi_{w}(z), \phi_{w}(w)) = d(z, w).
$$
(iii) We continue with the notations in part (ii). Assume that for a pair \((w_0, z_0)\) we have
\[
d(f(z_0), f(w_0)) = d(z_0, w_0).
\]
We define \(g = \phi f(w_0) \circ f \circ \phi^{-1} w_0\) just as before, and note that in the preceding inequalities we must be equalities throughout. Recall that equality occurs in Schwarz Lemma only if \(g\) is a rotation. But if \(g\) is a rotation, \(g\) is an automorphism of \(\Delta\). Then
\[
f = \phi^{-1} f(w_0) \circ g \circ \phi w_0
\]
is also an automorphism of \(\Delta\). By part (i), equality must hold for all pairs \(z, w\).

(iv) Applying \(\phi_s\) to all the arguments and by part (i), it is enough to show that
\[
d(z', 0) \leq d(w', 0) + d(z', w'),
\]
where \(z' = \phi_s(z)\) and \(w' = \phi_w(s)\). Thus we may assume \(s = 0\), and \(z, w\) arbitrary.

Note that \(d(u, 0) = |u|\). We thus need to show
\[
|z| \leq |w| + \frac{|z - w|}{1 - \bar{z}w} \iff |z| - |w| \leq \frac{|z - w|}{1 - \bar{z}w}.
\]
There are slick ways of solving this, but a straightforward solution is via polar coordinates. Set
\[
z = r_1 e^{it_1}, \quad w = r_2 e^{it_2}.
\]
We may rotate \(z, w\) to achieve \(z\) is positive real. Thus we may take \(t_1 = 0\) and \(t_2 = t\). We compute
\[
\frac{|z - w|}{1 - \bar{z}w} = \frac{r_1 - r_2 e^{it}}{1 - r_1 r_2 e^{it}}.
\]
The inequality to be proven becomes
\[
|r_1 - r_2 e^{it}| \geq |1 - r_1 r_2 e^{it}| |r_1 - r_2|.
\]
This follows by direct calculation. Squaring both sides, we obtain
\[
(r_1 - r_2 \cos t)^2 + (r_2 \sin t)^2 \geq (r_1 - r_2)^2 ((1 - r_1 r_2 \cos t)^2 + (r_1 r_2 \sin t)^2).
\]
This in turn becomes
\[
r_1^2 + r_2^2 - 2r_1 r_2 \cos t \geq (r_1 - r_2)^2 (1 + r_1^2 r_2^2 - 2r_1 r_2 \cos t)
\]
Fix \(r_1, r_2\). Regarding this as a linear inequality in \(\cos t\) which varies between \([-1, 1]\), we see that it suffices to check only the endpoints, when \(\cos t = \pm 1\).
(Think of two lines segments – how can you see they do not intersect?) When \(\cos t = 1\), the inequality to be proven becomes
\[
(r_1 - r_2)^2 \geq (r_1 - r_2)^2 (1 - r_1 r_2)^2
\]
which is true since
\[0 < 1 - r_1 r_2 \leq 1.\]

When \(\cos t = -1\), the inequality to be proven is
\[(r_1 + r_2)^2 \geq (r_1 - r_2)^2 (1 + r_1 r_2)^2.
\]

Assuming \(r_1 \geq r_2\) for convenience, this becomes
\[r_1 + r_2 \geq (r_1 - r_2)(1 + r_1 r_2) \iff 2r_2 \geq (r_1 - r_2) r_1 r_2 \iff 2 \geq (r_1 - r_2) r_1.
\]

This is indeed correct, as \(r_1 < 1\) and \(0 \leq r_1 - r_2 < 1\).

(v) Letting \(z = 1/3\) and \(w = 1/2\), we get (by part (ii))
\[|4a - 1| = d \left( a, \frac{1}{4} \right) \leq d \left( \frac{1}{3}, \frac{1}{2} \right) = \frac{1}{5},
\]
where \(a = f(1/3)\). Thus
\[|20a - 5| \leq |4 - a|.
\]

Using the triangle inequality, we see that
\[20|a| - 5 \leq |20a - 5| \leq |4 - a| \leq 4 + |a| \implies |a| \leq \frac{9}{19}.
\]

Similarly,
\[5 - 20|a| \leq |5 - 20a| \leq |4 - a| \leq 4 + |a| \implies \frac{1}{21} \leq |a|.
\]

Q2. Recall the Cayley transform which is a biholomorphism \(\phi : \Delta \to \mathbb{H}^+\) given by
\[\phi(w) = i \cdot \frac{1 - w}{1 + w}.
\]

To distinguish the notation, we denote \(d_\Delta\) and \(d_{\mathbb{H}^+}\) be the distance function defined on \(\Delta\) (in problem 1) and \(\mathbb{H}^+\) respectively.

Observe that for any \(z, w \in \Delta\),
\[d_{\mathbb{H}^+}(\phi(z), \phi(w)) = \frac{\left| \phi(z) - \phi(w) \right|}{\phi(z) - \phi(w)} = \frac{|1 - z| (1 + w) - (1 - w) (1 + z)}{|1 - z| (1 + w) + (1 - w) (1 + z)|} = \frac{|w - z|}{|1 - wz|} = d_\Delta(z, w).
\]

Therefore, the Cayley’s transformation \(\phi\) exchanges the two distances.

(i) If \(f\) is an automorphism of \(\mathbb{H}^+\), then
\[d_{\mathbb{H}^+}(z, w) = d_{\mathbb{H}^+}(f(z), f(w)).
\]

(ii) If \(f : \mathbb{H}^+ \to \mathbb{H}^+\) is a holomorphic function, then
\[d_{\mathbb{H}^+}(f(z), f(w)) \leq d_{\mathbb{H}^+}(z, w).
\]
The above two are restatement of problem 1 (i) and (ii): Note that for any automorphism (or holomorphic function) $f : h^+ \to h^+$, there is an automorphism (or holomorphic function) $g : \Delta \to \Delta$ such that $f = \phi \circ g \circ \phi^{-1}$. Thus the statements follow because the distance function is preserved under $\phi$. To be precise, when $f$ in case (ii)

$$d_{h^+}(f(z), f(w)) = d_{h^+}(\phi \circ g \circ \phi^{-1}(z), \phi \circ g \circ \phi^{-1}(w)) = d_{\Delta}(g \circ \phi^{-1}(z), g \circ \phi^{-1}(w)) \leq d_{\Delta}(\phi^{-1}(z), \phi^{-1}(w)) = d_{h^+}(z, w).$$

Q3. Consider the function $g : \Delta \to \Delta$ given by

$$g(z) = \begin{cases} \frac{f(z)}{r^n(z)} & \text{for } z \neq 0 \\ \frac{f^{(n)}(0)}{n!} & \text{for } z = 0 \end{cases}.$$  

The function $g$ is holomorphic i.e. the singularity at the origin is removable. This is guaranteed by the Taylor expansion for $f$. Indeed, we can write

$$f(z) = \sum_{k=n}^{\infty} a_k z^k$$

with radius of convergence at least 1, so the shifted power series

$$\frac{f(z)}{z^n} = \sum_{k=n}^{\infty} a_k z^{k-n}$$

also has radius of convergence at least 1 by the root test (or the ratio test). The sum must define a holomorphic function in $\Delta(0,1)$, which is obviously equal to $g$ at $z \neq 0$, while at $z = 0$ we obtain the value $a_n = \frac{f^{(n)}(0)}{n!}$.

Now fix $z \in \Delta$ and let $|z| < r < 1$. Note that for all $|w| = r$, we have

$$|g(w)| = \frac{|f(w)|}{|w^n|} \leq \frac{1}{r^n}.$$  

By the maximum modulus principle applied to $\Delta(0, r)$ we obtain

$$|g(z)| \leq \sup_{|w| \leq r} |g(w)| = \sup_{|w| = r} |g(w)| \leq \frac{1}{r^n}.$$  

Making $r \to 1$, we conclude

$$|g(z)| \leq 1.$$  

This means $|f(z)| \leq |z|^n$ for all $z$ (including $z = 0$ for trivial reasons), and

$$|g(0)| = \frac{|f^{(n)}(0)|}{n!} \leq 1 \implies |f^{(n)}(0)| \leq n!.$$  


Q4.

(a) Suppose $p$ and $q$ are two points in $U$ fixed by $f$. By Riemann mapping theorem, there exists a biholomorphism $\psi : U \to \Delta$. Consider the following commuting diagram

$$
\begin{array}{c}
U & \xrightarrow{\psi} & \Delta \\
\downarrow{f} & & \downarrow{g} \\
U & \xrightarrow{\psi} & \Delta \\
\end{array}
$$

where $g = \psi^{-1} \circ f \circ \psi : \Delta \to \Delta$, and horizontal arrows are biholomorphisms. Since $p, q$ are fixed for $f$, it follows that $\psi^{-1}(p)$ and $\psi^{-1}(q)$ are fixed for $g$. Indeed,

$$g(\psi^{-1}(p)) = \psi^{-1} \circ f \circ \psi \circ \psi(p) = \psi^{-1} f(p) = \psi^{-1}(p)$$

and similarly for $q$. We have shown in class that any automorphism of $\Delta$ has at most two fixed points, unless

$$g = 1 \implies \psi^{-1} \circ f \circ \psi = 1 \implies f = 1.$$

(b) Consider the entire function $f(z) = z^2$. The points 0 and 1 are fixed under $f$, and $f$ is clearly not the identity map.

(c) Let $U = \mathbb{C} \setminus \{0\}$, it is not simply connected (because integrating $1/z$ around the unit circle defies Cauchy's integral formula). Consider the function

$$f(z) = 1/z.$$ 

Note that $f$ is a holomorphic map from $U$ to $U$. Moreover, 1 and $-1$ are the fixed points since $f(-1) = -1$ and $f(1) = 1$.

Q5. Refer to Figure 0.1. We claim that the function

$$f(z) = \frac{e^{iz/2} - 1}{e^{iz/2} + 1}$$
Figure 0.2. Biholomorphic map from $\Delta \setminus (-1, 0]$ to $\Delta$.

is a biholomorphic map from $\{z : -1 < \text{Re} z < 1\}$ to $\Delta$. This follows by noting that $f$ is the composition of $g_1 \circ g_2 \circ g_3$ where the following are biholomorphisms:

$$g_1(z) = \frac{z - 1}{z + 1} : \{\text{Re} z > 0\} \to \Delta$$

$$g_2(z) = e^z : \{\frac{-\pi}{2} < \text{Im} z < \frac{\pi}{2}\} \to \{\text{Re} z > 0\}$$

$$g_3(z) = \frac{i\pi z}{2} : \{z : -1 < \text{Re} z < 1\} \to \{\frac{-\pi}{2} < \text{Im} z < \frac{\pi}{2}\}.$$ 

Q6. Refer to Figure 0.2. We claim that the function

$$f(z) = \frac{(\sqrt{z} - i)^2 - i(\sqrt{z} + i)^2}{(\sqrt{z} - i)^2 + i(\sqrt{z} + i)^2}$$

is a biholomorphic map from $\Delta \setminus (-1, 0]$ to $\Delta$. This follows by noting that $f$ is the composition of $g_1 \circ g_2 \circ g_3 \circ g_4$ where the following are biholomorphisms:

$$g_1(z) = \frac{z - i}{z + i} : \{\text{Im} z > 0\} \to \Delta$$

$$g_2(z) = z^2 : \{\text{Im} z > 0, \text{Re} z > 0\} \to \{\text{Im} z > 0\}$$

$$g_3(z) = \frac{2 - i}{z + i} : \{|z| < 1, \text{Re} z > 0\} \to \{\text{Im} z > 0, \text{Re} z > 0\}$$

$$g_4(z) = \sqrt{z} : \Delta \setminus (-1, 0] \to \{|z| < 1, \text{Re} z > 0\}.$$ 

The square root map $g_4$ is defined by taking the standard branch of logarithm and defining

$$\sqrt{z} = e^{\frac{1}{2} \log(z)}.$$ 

The map $g_3$ is a biholomorphism because it is the Möbius transformation sending the line $y = 0$ and the circle $|z| = 1$ to the lines $x = 0$ and $y = 0$. The the region enclosed are mapped biholomorphically.