Math 220B, Problem Set 1. Due Tuesday, January 16.

1. (*Partition Products.*) Euler studied the products

$$Q(z) = \prod_{n=1}^{\infty} (1 + q^n z)$$

in connection with the theory of partitions and pentagonal numbers $\frac{n(3n-1)}{2}$. These products thus appear in *combinatorics* as well as *number theory.*

Remark: For fixed values of z, we can study the power series expansion of Q viewed as function of q. Two cases are of special interest. For $z = 1$, one easily sees that

$$Q(1) = \prod_{n=1}^{\infty} (1 + q^n) = \sum_{n=0}^{\infty} p(n) q^n$$

where $p(n)$ is the number of partitions into distinct parts. When $z = -1$, Euler's pentagonal number theorem states that

$$Q(-1) = \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}}.$$

We will not need/prove these statements here. Instead, our point of view will be to regard Q as a function of z, for fixed $|q| < 1$.

(i) Show that Q is an entire function in z.

(ii) Show that $Q(z) = (1 + qz)Q(qz)$.

(iii) Write

$$Q(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Derive the recursion

$$a_n = \frac{q^n}{1 - q^n} a_{n-1}$$

and derive the identity

$$Q(z) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)} z^n.$$

(iv) Write out the strange looking identities proved in (iii) for $z = 1$ and $z = -1$.

Remark: Digressing further, in number theory, there are several similar looking (but harder) identities. An example is the Rogers-Ramanujan identity

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

The combinatorial consequence is that the number of partitions of n whose parts differ by at least 2 (whose generating series can be shown to be the expression on the left) equals the number of partitions of n whose parts are congruent to $\pm 1 \mod 5$ (whose generating series is the expression on the right).

2. (*Towards the Γ-function.*) Write Log for the principal branch of the logarithm.

(i) Possibly using Taylor expansion, show that if $|w| \leq \frac{1}{2}$,

$$|\mathrm{Log}(1+w) - w| \leq 2|w|^2.$$

(ii) Let $a, b \in \mathbb{C}$ have positive real parts. Show that

$$\mathrm{Log}(ab) = \mathrm{Log}(a) + \mathrm{Log}(b).$$

(iii) Let $r > 0$. Show that there exists N such that for all $n \geq N$, $1 + \frac{z}{n}$ and $e^{-\frac{z}{n}}$ have positive real parts for all $z \in \Delta(0, r)$.

(iv) (Taking Logs and arguing that the series of Log's coverges absolutely and locally uniformly), show that the product

$$G(z) = \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right)$$

converges to an entire function.

3. (*Blaschke products.*) This is a modified version of Conway VII.5, Problem 4. The point of this question is to characterize certain holomorphic functions on the unit disc.

For $\alpha \in \Delta(0,1) \setminus \{0\}$, define

$$B_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha}.$$

(i) Let $\alpha \in \Delta(0,1) \setminus \{0\}$ and $z \in \overline{\Delta}(0, r)$ for $r < 1$. Prove that

$$\left| \frac{\alpha + |\alpha| z}{(1 - \bar{\alpha} z)\alpha} \right| \leq \frac{1+r}{1-r}.$$

Show that on $\overline{\Delta}(0, r)$, we have

$$|1 - B_\alpha(z)| \leq \frac{1+r}{1-r}(1 - |\alpha|).$$

(ii) Let $\alpha_n \in \Delta(0,1) \setminus \{0\}$ be a sequence of nonzero numbers in the unit disc. Assume that

$$\sum_n (1 - |\alpha_n|) < \infty.$$

Using (i), show that the *Blaschke product*

$$B(z) = \prod_{n=1}^{\infty} B_{\alpha_n}(z)$$

converges to a holomorphic function $B : \Delta(0,1) \to \mathbb{C}$ with zeroes only at α_n.

Remark: Conversely, it can be shown using material from Math 220C (Jensen's formula) that the zeroes of a bounded holomorphic function B on $\Delta(0,1)$ satisfy

$$\sum_n (1 - |\alpha_n|) < \infty.$$

Parts (iii) and (iv) concern finite Blaschke products, and can be solved using only the material from Math 220A. Part (iv) can be viewed as a generalization of Question 6 on the Final Exam for Math 220A.

(iii) Show that for $\alpha \in \Delta(0,1) \setminus \{0\}$,

$$B_\alpha : \overline{\Delta}(0,1) \to \overline{\Delta}(0,1)$$

in such a fashion that

$$|B_\alpha(z)| = 1 \text{ for } |z| = 1.$$

(iv) Conversely, let $f : \Delta(0,1) \to \Delta(0,1)$ be a holomorphic function extending continuously to $\overline{\Delta}(0,1)$ such that

$$|f(z)| = 1 \text{ for } |z| = 1.$$

Show that f can be expressed as a finite Blaschke product:

$$f(z) = cz^m \prod_{n=1}^{N} B_{\alpha_n}(z).$$

Hint: Assume first that $f(0) \neq 0$. Show that f can have only finitely many zeroes $\alpha_1, \ldots, \alpha_n \in \Delta(0,1) \setminus \{0\}$. Construct the suitable Blaschke product $B(z)$ and work with the function $g(z) = f(z)/B(z)$. What properties does g have?

Finally, we give a few quick applications to questions from past Qualifying Exams.

(v) (*Qualifying Exam, Spring 2018.*) Give an example of a holomorphic function $f : \Delta(0,1) \to \mathbb{C}$ with simple zeros only at $\alpha_n = 1 - \frac{1}{n^2}$, for $n \geq 1$.

(vi) (*Qualifying Exam, Fall 2021.*) If $f : \mathbb{C} \to \mathbb{C}$ is an entire function with $|f(z)| = 1$ for all $|z| = 1$, show that $f(z) = cz^n$ for some $c \in \mathbb{C}$, $n \geq 0$.

Remark: You should solve this question as an application of (iv). There are other ways of solving the question without (iv).

(vii) (*Qualifying Exam, Fall 2020.*) Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ with $|f(z)| = 2$ for $|z| = 1$ and $f^{(3)}(0) = -12$.

4. (*Functions with the same zeros. This only requires material from Math 220A.*)

(i) Show that if $f, g : U \to \mathbb{C}$ are two holomorphic functions in a simply connected region that have the same zeros with the same multiplicity, then there exists a holomorphic function $h : U \to \mathbb{C}$ such that $f = e^h g$.

Hint: Construct a logarithm for f/g using Lecture 6, Math 220A.

(ii) Show that the conclusion of (i) is false without the assumption that U is simply connected.