Math 220, Problem Set 2. Due Tuesday, January 30.

This is a longer problem set, so you may wish to start early and solve the questions as we progress.

1. (*Applications of the* Γ*-function.*) Let

$$R(z) = \frac{P(z)}{Q(z)}$$

be a rational function without poles at the positive integers.

Remark: To put things into perspective, recall that Problem 4, Problem Set 7 in Math 220A gave a formula for the infinite sum $\sum_{k=-\infty}^{\infty} R(k)$ under certain assumptions. In this problem, we study the infinite product $\prod_{k=1}^{\infty} R(k)$.

Assume that P, Q are polynomials of the same degree, with leading term equal to 1. Write

$$P(z) = \prod_{i=1}^{d}(z - a_i), \quad Q(z) = \prod_{i=1}^{d}(z - b_i)$$

and furthermore assume that

$$\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i.$$

(One can see that without these assumptions the product does not converge.)

(i) Using the definition of the function G, show that

$$\prod_{k=1}^{\infty} R(k) = \frac{G(-a_1)\cdots G(-a_d)}{G(-b_1)\cdots G(-b_d)}.$$

Express the product $\prod_{k=1}^{\infty} R(k)$ in terms of the Γ function.

(ii) Using (i), compute the product

$$\prod_{n=1}^{\infty} \frac{n^2 + n - 4/9}{n^2 + n - 5/16}.$$

You can simplify the answer so that only sine's are involved.

For the next two questions, you will only need to know that the Weierstraß problem admits a solution. (Feel free to use the explicit form of the solution if it helps you, though this is not strictly speaking needed.)

2. (*Greatest common divisor.*) This is Conway VII.5, Problem 3. Assume that f and g are entire functions. Show that there exist entire functions h, F and G such that

$$f(z) = h(z)F(z), \; g(z) = h(z)G(z)$$

with F, G having no common zeroes.

3. (*Roots.*) Let f be an entire function and $n \geq 1$. Show that there exists an entire function g such that $g^n = f$ if and only if the orders of all zeroes of f are divisible by n.

The next question introduces some new functions that were not part of the traditional arsenal of examples in Math 220A.

4. (*The Weierstraß σ and ζ functions.*) This is a modified version of Conway VII.5, Problem 13 and Conway VIII.3, Problem 2.

Let ω_1, ω_2 be two non-zero complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$. Let

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

Part A. Solve the Weierstraß problem of finding an entire function with simple zeroes at the points in the lattice Λ.

To this end, define the Weierstraß σ-function as the infinite product

$$\sigma(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{1}{2} \cdot \frac{z^2}{\lambda^2}\right) = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right).$$

(i) Show that $\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3}$ converges.

Hint: Show that there exists a number $c > 0$ such that

$$|n\omega_1 + m\omega_2| \geq c(|n| + |m|),$$

for all real numbers n, m. Show that the number of integer solutions of $|n|+|m| = k$ is equal to $4k$. Conclude that

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3} \leq 4c^{-3} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

(ii) Show that σ is an entire function with simple zeroes only at the points of Λ.

Remark: The σ function is important in the study of elliptic functions (see Math 220A, Lecture 16 for the definition). In fact, all elliptic functions can be expressed in terms of σ in a very precise way. The function σ also appears in the study of Riemann surfaces of genus 2.

Part B. A Mittag-Leffler problem for Λ.

(iii) Weierstraß also defined the function ζ (not to be confused with Riemann's zeta) by taking logarithmic derivatives

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.$$

Show that

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right).$$

Thus, the Weierstraß ζ function solves a Mittag-Leffler problem for Λ. Which one?

(This doesn't require much. We are just making the connection between the functions σ and ζ.)

(iv) Let a_n be a sequence of nonzero distinct complex numbers with $a_n \to \infty$. Can you generalize (iii) to a statement relating the solution to the Weierstraß Factorization for the set $\{a_n\}$ to the solution to a certain Mittag-Leffler problem for $\{a_n\}$? Your statement should involve the logarithmic derivative.

(You only need to formulate the statement and show why it holds via a short computation.)

5. (*The Weierstraß $\wp$-function. This uses only the methods of Math 220A.*) Probably the best known Weierstraß function is the function *"pe"* obtained from the derivative of the function ζ:

$$\wp(z) = -\zeta'(z).$$

(i) By direct calculation, show that

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

(There is no need to verify local uniform and absolute convergence of the series, this follows for free from the theorems on taking derivatives we have developed and the fact that the product giving σ converges absolutely locally uniformly.)

(ii) Show that $\wp$ is an elliptic function in the sense defined in Math 220A, Lecture 16. That is, show that $\wp$ is meromorphic and doubly periodic

$$\wp(z) = \wp(z + \omega_1) = \wp(z + \omega_2).$$

(iii) Using (ii), show that $\wp'$ is doubly periodic as well.

(iv) Using the definition, show that $\wp$ is even.

(v) Show that the Laurent expansions around 0 of $\wp$ and $\wp'$ take the form

$$\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + \dots$$

$$\wp'(z) = -\frac{2}{z^3} + 2az + 4bz^3 + \dots$$

for certain $a, b \in \mathbb{C}$.

(vi) Using undetermined coefficients show that there exist constants A, B such that

$$\wp'^2 - (4\wp^3 + A\wp + B)$$

has no Laurent principal tail at $z = 0$ and vanishes at $z = 0$.

(vii) Conclude from (vi) and double periodicity that

$$\wp'^2 - (4\wp^3 + A\wp + B) = 0.$$

Remark: The function $\wp$ is truly very interesting. There are connections between $\wp$ (naturally arising in *complex analysis*) and *algebraic geometry* (cubic/elliptic curves), *number theory* (modular forms), and *mathematical physics* that one may not have guessed at first:

(a) $\wp$ and $\wp'$ parametrize the cubic (elliptic) curves

$$y^2 = 4x^3 + Ax + B$$

for suitable A, B via $y = \wp'(z), x = \wp(z)$.

The expressions A, B depend on ω_1, ω_2 and are called Eisenstein series. They are examples of modular forms.

(b) using (a), one can show that $\wp$ provides a solution for the equation of waves in shallow waters (Korteweg-de Vries equation):

$$\wp''' = 12\wp\wp'.$$

6. (*The generalized Weierstraß problem.*) Let $\{a_n\}$ be distinct complex numbers with $a_n \to \infty$. Fix complex numbers $\{A_n\}$. Show that there exists an entire function f such that

$$f(a_n) = A_n.$$

This is a special case of Conway VIII.3.5, page 209, which you may wish consult for hints.

Remark: In fact, it is possible to prescribe the values of the derivatives

$$f^{(k)}(a_n) = A_{nk}$$

for $0 \leq k < m_n$, where m_n are fixed positive integers and A_{nk} are arbitrary complex numbers, $0 \leq k < m_n$. The proof of this assertion *does not* require new ideas, it is just notationally more involved.