Math 220, Problem Set 3.

1. (*Qualifying Exam, Spring 2017.*)

(i) Construct an entire function with simple zeroes only at $A = \left\{\sqrt{n} + \frac{1}{\sqrt{n}}, n = 1, 2, \ldots\right\}$ and no other zeroes. Please justify your answer.

(ii) Write down a meromorphic function in $\mathbb{C}$ only with simple poles at $z = n$ and residues equal to $\sqrt{n}$ for each $n = 1, 2, \ldots$. Please justify your answer.

2. Let f be an entire function and let $\mathcal{F}$ denote the family of functions $f(kz)$ for $k \in \mathbb{Z}_{>0}$ defined in the annulus $r_1 < |z| < r_2$ for $r_1, r_2 > 0$. Show that $\mathcal{F}$ is normal iff f is constant.

3. (*Qualifying Exam, Spring 2017. Also on Qualifying Exam, Spring 2020 and Qualifying Exam, Fall 2013 in slightly different formulations.*)

Let $\mathcal{F}$ be the family of holomorphic functions in $\Delta(0,1)$ with $f(0) = 1$ and $\text{Re } f > 0$. Show that $\mathcal{F}$ is normal.

Hint: You may wish to remember the Cayley transform from Math 220A.

4. (*Vitali's theorem.*) Prove Vitali's theorem, Conway VII.2.4, page 154. The statement is as follows.

Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions in $U \subset \mathbb{C}$, and let f be holomorphic in U. If
$$A = \{z \in U : f_n(z) \to f(z)\}$$
has a limit point in U, then f_n converges locally uniformly to f in U.

5. (*Normal families under composition.*) Solve Conway VII.2.7, page 154.

Let $U, \Omega \subset \mathbb{C}$ be open connected. Let $g : \Omega \to \mathbb{C}$ be a holomorphic function which is bounded on bounded sets. Let $\mathcal{F}$ be a normal family of holomorphic functions $f : U \to \Omega$. Show that the family $\{g \circ f : f \in \mathcal{F}\}$ is also normal.

6. (*Qualifying Exam, Spring 2012.*) Solve the following version of Conway VII.2.8, page 154. The statement is as follows.

Let $\mathcal{F}$ be a family of holomorphic functions in $\Delta(0,1)$. Assume there exist $M_n > 0$ constants with
$$\limsup M_n^{\frac{1}{n}} \leq 1$$
such that for all $f \in \mathcal{F}$,
$$\frac{|f^{(n)}(0)|}{n!} \leq M_n.$$
Show that $\mathcal{F}$ is normal. Show that the converse is also true.