Math 220, Problem Set 4.

1. (*Qualifying Exam, Fall 2017.*) Let $\mathcal{F}$ be the family of holomorphic functions $f : \mathfrak{h}^+ \to \mathbb{C}$ such that $f(i) = 0$ and $|f(z)| < 1$ for all $z \in \mathfrak{h}^+$. Find the maximum value of $|f(2i)|$ for $f \in \mathcal{F}$.

2. (*Generalized Schwarz Lemma.*) Assume $f : \Delta \to \Delta$ is holomorphic with a zero of order n at the origin. Show that

$$|f(z)| \leq |z|^n \quad \forall z \in \Delta, \qquad \text{and} \qquad |f^{(n)}(0)| \leq n!.$$

Hint: This is very similar to the proof of Schwarz lemma given in class.

3. (*Holomorphic maps between punctured discs. Qualifying Exam, Spring 2020.*) Let $a, b \neq 0$ and $a, b \in \Delta(0,1)$. Consider the twice punctured discs

$$D_1 = \Delta(0,1) \setminus \{0, a\}, \quad D_2 = \Delta(0,1) \setminus \{0, b\}.$$

Find a necessary and sufficient condition for D_1, D_2 to be biholomorphic, and determine all biholomorphic maps

$$f : D_1 \to D_2.$$

Hint: This is very similar to the proof given in class for the automorphisms of the punctured unit disc $\Delta \setminus \{0\}$.

Remark: When $a = b$, $a \neq 0$, your answer will imply that the automorphism group of $\Delta \setminus \{0, a\}$ has 2 elements, hence it is $\mathbb{Z}/2\mathbb{Z}$.

4. (*Schwarz-Pick for the unit disc and a bit of hyperbolic geometry.)* The pseudo-hyperbolic distance on the unit disc $\Delta = \Delta(0,1)$ is given by

$$d(z,w) = \left| \frac{z-w}{1-\bar{z}w} \right|, \quad z, w \in \Delta.$$

(i) Show, by direct calculation, that if f is an automorphism of Δ then

$$d(z,w) = d(f(z), f(w)).$$

In other words, automorphisms of Δ preserve the pseudo-hyperbolic distance i.e. they are *isometries.*

Hint: You may wish to recall that f is a composition of a rotation with the fractional linear transformation ϕ_a. It thus suffices to check the above equality for f a rotation and for $f = \phi_a$ separately. The last case requires an explicit calculation.

(ii) Let $f : \Delta \to \Delta$ be holomorphic. Using (i) to reduce to a familiar case, show that

$$d(f(z), f(w)) \leq d(z,w).$$

Thus holomorphic maps contract the pseudo-hyperbolic distance.

Hint: Recentering (i.e. considering $\Phi \circ f \circ \Psi$ for suitable automorphisms Φ, Ψ of the disc) you may assume $f(w) = 0$ and $w = 0$. This case should be familiar.

(iii) Show that if there exist $z, w \in \Delta$ and $f : \Delta \to \Delta$ holomorphic with

$$d(f(z), f(w)) = d(z, w)$$

then f is an automorphism of Δ. Consequently, by (i), the equality

$$d(f(z), f(w)) = d(z, w)$$

holds for all $z, w \in \Delta$.

Remark: As a corollary, automorphisms of Δ are exactly the holomorphic maps that are isometries with respect to d.

(iv) *Optional:* show that d is indeed a distance. That is, show that

$$d(z, s) \leq d(w, s) + d(z, w).$$

You may wish to reduce to the case $s = 0$ using part (i).

Hint: When $s = 0$, you may first rotate z to make it positive real. Using polar coordinates $z = r_1, w = r_2 e^{it}$ you need to establish a linear inequality in $\cos t$ which only needs to be checked at the endpoints $\cos t = \pm 1$ (why?)

(v) As an application to (ii), assume $f : \Delta \to \Delta$ satisfies $f\left(\frac{1}{2}\right) = \frac{1}{4}$. Show that

$$\frac{1}{21} \leq \left| f\left(\frac{1}{3}\right) \right| \leq \frac{9}{19}.$$

Remark (only if you have seen some differential geometry): The hyperbolic distance is given by $2 \tanh^{-1} d(z, w)$. It comes from the metric

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

on the unit disc Δ, whose Gaussian curvature equals -1. The Schwarz-Pick lemma can be further generalized with this observation as the starting point, for holomorphic maps between domains/Riemann surfaces with appropriate curvature.

5. (*Qualifying Exam, Spring 2021.*) Let $f_n : \Delta \to \Delta$ be a family of automorphisms of Δ converging locally uniformly to a nonconstant function f. Show that f is an automorphism of Δ.

Hint: Problem 4 (i) (iii). You can also solve this as an application of Montel, but the solution is more difficult.

6. (*Schwarz-Pick for the upper half plane.*) On the upper half plane, define

$$d(z, w) = \left| \frac{z - w}{z - \overline{w}} \right|, \quad z, w \in \mathfrak{h}^+.$$

Formulate and briefly justify the analogues of Problem 4 (i) and (ii) for $\mathfrak{h}^+$.

Hint: You won't have to redo the proofs in Problem 4 – the shortcut is to check that the Cayley transform exchanges the two distances defined in Problem 4 and Problem 6.