Math 220, Problem Set 5.

1. (*Qualifying Exam, Fall 2022, expanded version. Also on Qualifying Exam, Fall 2008.*) Let $U \subset \mathbb{C}$ be simply connected region, $U \neq \mathbb{C}$.

(i) Show that any holomorphic map $f : U \to U$, other than the identity, admits at most one fixed point.
(ii) Show that the above statement is false for $U = \mathbb{C}$.
(iii) Exhibit a nonsimply connected set U and a function $f : U \to U$, not equal to the identity, for which (i) fails.

For the next few questions, it helps to think in terms of simple geometric moves. The examples to be constructed involve compositions of familiar functions (rotations, exponentials, logarithms, squaring, square roots, Cayley transform, etc), appropriately modified.

2. (*Riemann Mapping Theorem.*) Exhibit a biholomorphic map from $\{z : -1 < \text{Re } z < 1\}$ to the unit disc.

3. (*Riemann Mapping Theorem.*) Let $\Delta^+ = \Delta \cap \mathfrak{h}^+$ denote the upper half unit disc, and let Δ^- denote the lower half disc. Construct a biholomorphism Δ^+ to Δ as follows.

(i) Let $C^{-1} : \Delta \to \mathfrak{h}^+$ denote the inverse Cayley transform
$$C^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$
Compute the images of $-1, 0, 1, -i, +i$ under C^{-1}. How does C^{-1} transform the boundaries of Δ^+ and Δ^-?
(ii) Deduce that C^{-1} maps Δ^+ and Δ^- to the second and first quadrant respectively.
(iii) Construct a biholomorphism between the upper half disc Δ^+ and Δ.

4. (*Riemann Mapping Theorem.*) Exhibit a biholomorphic map between the slit unit disc $\Delta \setminus (-1, 0]$ and the unit disc.

5. (*Qualifying Exam, Fall 2022, with some hints.)* Let $f : \mathbb{C} \to \mathbb{C}$ be entire such that $|f(z)| = 1$ for all z real.

(i) Show that $f(z)\overline{f(\overline{z})} = 1$ for all $z \in \mathbb{C}$.
(ii) Show that there exists an entire function g such that $f(z) = e^{2\pi i g(z)}$.
(iii) Describe all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that $|f(z)| = 1$ if z is real.

6. (*Schwarz Reflection across arcs.)* Solve Conway, IX.1.2, page 213. The (slightly) modified statement is as follows.

Let $U \subset \mathbb{C}$ be an open set outside of the unit disc whose boundary shares an arc with the unit circle. Define
$$U^* = \{z : 1/\bar{z} \in U.\}.$$
The set U^* is the reflection of U across the unit circle $|z| = 1$.

(i) If $U = \{1 < |z| < R\}$, what is U^*?

(ii) Show that U^* is an open subset of $\Delta \setminus \{0\}$.

(iii) Let $f : U \to \mathbb{C}$ be holomorphic and nowhere zero, and define $f^*(z) = 1/\overline{f\,(1/\bar{z})}$. Show that f^* is holomorphic in U^*.

(iv) What would it mean for an open set V to be symmetric with respect to an arc of the unit circle?

(v) Formulate and prove a version of Schwarz reflection where the unit circle $|z| = 1$ replaces the real axis (both in the domain and the target of your function).

Perhaps the easiest proof is to use the Cayley transform.