Math 220, Problem Set 6.

1. Let $K = \left\{z : \frac{1}{4} \le |z| \le \frac{3}{4}\right\}$, $\Delta = \Delta(0,1)$. Show that there exists a function which is holomorphic in K, and which cannot be approximated uniformly in K by holomorphic functions in Δ.

2. (*Qualifying Exam, Spring 2023, slightly modified.*) Let $f(z) = \frac{1}{(z-2)(z-6)}$.

(i) Does there exists a sequence of rational functions R_n with poles only at 3 and 7 such that

$$\lim_{n\to\infty} \sup_{4\le|z|\le5} |f(z) - R_n(z)| = 0?$$

(ii) Does there exist a sequence R_n of rational functions as above, but with poles only at 7?

3. (*Pointwise limit of holomorphic functions may not be holomorphic.*) Show that there exist polynomials p_n such that the pointwise limit

$$\lim_{n\to\infty} p_n(z) = \begin{cases} 1 & \text{if } z \in \mathfrak{h}^+ \\ 0 & \text{if } z \in \mathbb{R} \\ -1 & \text{if } z \in \mathfrak{h}^-. \end{cases}$$

(i) Let

$$K_n = \{z = x + iy : \frac{1}{n} \le |y| \le n,\ |x| \le n\} \cup \{z \in \mathbb{R}, |z| \le n\}.$$

Note that K_n is compact. Write

$$K_n = K_n^+ \cup K_n^- \cup K_n^0$$

for the intersections with $\mathfrak{h}^+, \mathfrak{h}^-, \mathbb{R}$. Consider the function

$$f_n = \begin{cases} 1 & \text{if } z \in K_n^+ \\ 0 & \text{if } z \in K_n^0 \\ -1 & \text{if } z \in K_n^-. \end{cases}$$

Show that f_n extends holomorphically to a neighborhood of K_n. Show there exist polynomials p_n such that

$$|f_n - p_n| < \frac{1}{2^n} \text{ in } K_n.$$

(ii) Conclude that the polynomials p_n satisfy the above property.

Remark: It is easy to construct sequences of continuous functions whose pointwise limit is discontinuous. It is not as easy to construct sequences of holomorphic functions whose pointwise limit is not holomorphic. An example is provided by this question.

4. (*Qualifying Exam, Spring 2021.*) Let K be a proper closed arc of the unit circle $|z| = 1$.

(i) Is there a sequence of polynomials $P_n(z)$ such that $P_n(z) \to \bar{z}$ uniformly in K?

(ii) Is there a sequence of polynomials $P_n(z)$ such that $P_n(z) \to \bar{z}$ uniformly on the circle $|z| = 1$?

5. (*Qualifying Exam, Fall 2021.*) Let $K = \{z \in \mathbb{C} : |z| \leq 3,\ |z-1| \geq 1,\ |z+1| \geq 1\}$.

(i) True/false: every holomorphic function in a neighborhood of K is the local uniform limit on K of a sequence of polynomials. Please justify your answer.

(ii) Determine, with justification, the set

$$\widehat{K} = \{z \in \mathbb{C} : |p(z)| \leq \sup_{w \in K} |p(w)| \text{ for all polynomials } p\}.$$

6. Let $U = \left\{z : |z| < 1, \left|z - \frac{1}{4}\right| > \frac{3}{4}\right\}$, $K = \left\{z : |z| \leq 1, \left|z - \frac{1}{4}\right| \geq \frac{3}{4}\right\}$. True or false (please justify):

(i) Every continuous function in K which is holomorphic in U is uniform limit in K of a sequence of polynomials.

(ii) Every holomorphic function in K can be approximated uniformly in K by Laurent polynomials. A Laurent polynomial is an expression of the form

$$\sum_{n=-N}^{N} a_n z^n.$$

(iii) Every holomorphic function in U can be approximated locally uniformly in U by polynomials.