Math 220B - Winter 2024 - Final Exam

Name: ______________________________

Student ID: ___________________________

Instructions:

There are 7 questions which are worth 70 points.

You may not use any books, notes, internet. If you use a homework problem you will need to reprove it.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

Problem 1. [*10 points.*]

Construct a biholomorphism between the strip $S = \{z = x + iy : 0 < x + y < \pi\}$ and the unit disc $\Delta(0, 1)$.

Problem 2. [*10 points.*]

Let $U \subset \mathbb{C}$ be an open connected set. Show that the following statements are equivalent:

(i) U is simply connected

(ii) for all holomorphic functions $f, g : U \to \mathbb{C}$ such that $f^2 + g^2 = 1$, there exists a holomorphic function $h : U \to \mathbb{C}$ such that

$$f = \cos h, \quad g = \sin h.$$

Hint: You may wish to observe that $f^2 + g^2 = (f + ig)(f - ig) = 1$.

Problem 3. [*10 points; 3, 7.*]

Let $a_1, \ldots, a_n$ be n complex numbers with $|a_k| < 1$ for all $1 \leq k \leq n$. Consider the product

$$f(z) = \frac{z - a_1}{1 - \bar{a}_1 z} \cdots \frac{z - a_n}{1 - \bar{a}_n z}.$$

(i) Using results proven in class, briefly explain that $f : \Delta \to \Delta$ is holomorphic and $|f(z)| = 1$ for all $|z| = 1$.

(ii) Let $|\lambda| < 1$. Show that there exist n complex numbers $b_1, \ldots, b_n$ with $|b_k| < 1$ for $1 \leq k \leq n$, and $|\mu| = 1$ such that

$$\frac{f(z) - \lambda}{1 - \overline{\lambda} f(z)} = \mu \cdot \frac{z - b_1}{1 - \overline{b}_1 z} \cdots \frac{z - b_n}{1 - \overline{b}_n z}.$$

Problem 4. [*10 points.*]

Let $f : \Delta(0,1) \to \mathbb{C}$ be such that $\operatorname{Re} f(z) > 0$ for all $z \in \Delta(0,1)$, and assume that $f(0) = 1$. Show that for all $z \in \Delta(0,1)$ we have

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

Problem 5. [*10 points; 3, 3, 4.*]

Let $U \subset \mathbb{C}$ be a connected open set containing 0. Let $f : U \to U$ be a holomorphic function such that $f(0) = 0$.

(i) Let

$$f_n = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}$$

denote the iterates of f. Compute $f_n'(0)$ in terms of $f'(0)$.

(ii) Show that if $|f'(0)| > 1$ there exists no open subset $0 \in V \subset U$ such that the family

$$\mathcal{F} = \{f_n : n \geq 1\}.$$

is normal on V.

(iii) Show that if U is bounded, then $|f'(0)| \leq 1$.

Problem 6. [*10 points; 2, 4, 4.*]

Let $f : \Delta \to Q$ be a biholomorphism between the unit disc and the square

$$Q = \{z \in \mathbb{C} : -1 < \text{Re } z < 1, \ -1 < \text{Im } z < 1\},$$

such that $f(0) = 0$. Show that

$$f(iz) = if(z).$$

(i) Show that $g : \Delta \to Q$, $g(z) = if(z)$ is also a biholomorphism between Δ and Q.

(ii) Show that there exists a rotation $R : \Delta \to \Delta$ such that

$$g = f \circ R.$$

(iii) Show that $R(z) = iz$ and conclude.

Problem 7. [*10 points; 5, 5.*]

(i) Construct compact (possibly disconnected) sets $K_n \subset \mathbb{C}$, $n \geq 1$, such that

(1) $\bigcup_{n=1}^{\infty} K_n = \mathbb{C} \setminus \{0\}$

(2) $K_n \subset K_{n+1}$

(3) $\mathbb{C} \setminus K_n$ is connected.

(ii) Show that there exist polynomials $p_n(z)$ such that

(1) $p_n(0) = 1$

(2) $p_n(z) \to 0$ for all $z \in \mathbb{C} \setminus \{0\}$ (pointwise convergence).

Hint: $f(z) = 1/z$.