Math 220B - Lecture 1

January 8, 2024

0. Logistics

(1) Mostly as last quarter.

(2) Office Hour M 3:30 – 5 PM

(3) PSets – due "Tuesdays" weekly

(4) Grades – 30% HWK

30% midterm

40% Final

(5) Midterm – February 12, in class

(6) Final – March 18, 11:30 – 2:30

(7) Canvas / Gradescope / Website

math.ucsd.edu/~dopra/220w24.html

III Topics to be covered

Part I : Sequences / Series / Products

(1) infinite products of holomorphic functions

Weierstraß Problem

(2) sequences & series of meromorphic functions

Mittag-Leffler Problem

(3) sequences of hol functions, Montel families

Part II : Geometric aspects / Conformal maps

(4) Schwarz Lemma, automorphisms of $\Delta, \mathfrak{H}^+, \Delta^*, \ldots$

(5) Riemann mapping theorem

Part III. Further topics (if time)

(6) Runge's theorem

(7) Schwarz Reflection

[2] Three Motivating Questions for Part I

Math 220A, Lecture 8 : $f \not\equiv 0$ entire has countably many zeroes that do not accumulate.

Weierstraß Problem

Given a sequence of distinct $\{a_n\}$, $a_n \to \infty$ and positive integers $\{m_n\}$, is there an entire function with zeroes only at $\{a_n\}$ with order exactly $\{m_n\}$. ?

Weierstraß$^+$ Problem

Given $\{a_n\}$, $\{m_n\}$ as above, $\{A_{nj}\}_{0 \le j < m_n}$ is there an entire function f with

$$f^{(j)}(a_n) = A_{nj} \quad \forall \; 0 \le j < m_n$$

Mittag - Leffler Problem

Take $\{a_n\}$ as above.

We can always find a meromorphic function f in $\mathbb{C}$ with poles only at a_n. e.g. take g solving Weierstraß at $\{a_n\}$ and set $f = 1/g$.

Mittag - Leffler asks if we can furthermore prescribe the Laurent principal parts.

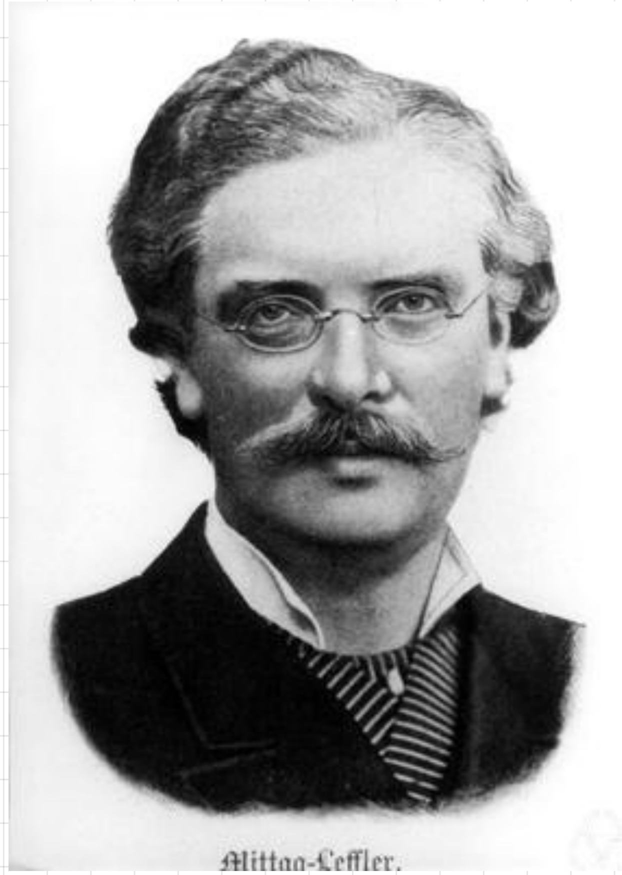
Given $\{a_n\}$ distinct, $a_n \to \infty$, and polynomials $p_n\left(\frac{1}{z-a_n}\right)$ without constant terms, is there a meromorphic function in $\mathbb{C}$ with poles only at a_n and Laurent expansion

$$f = p_n\left(\frac{1}{z-a_n}\right) + \dots \quad \text{near } a_n.$$

Weierstraß – Poincaré Problem

Is any meromorphic function a quotient of two holomorphic functions?

Remark The three questions above can be asked & answered for all $\mathcal{U} \subseteq \mathbb{C}$ open & connected.



Karl Weierstraß

1815 – 1897

Gösta Mittag-Leffler

1846 – 1927

Tools — sequences, series, products of

holomorphic & meromorphic functions.

Last quarter [i] sequences

of holomorphic functions.

[ii] series

This quarter

[i] Weierstraß requires infinite products

Naively, $f(z) = \prod_n \left(1 - \frac{z}{a_n}\right)$ but ...

does it converge?

[ii] Mittag - Leffler requires infinite sums of meromorphic functions.

Naively, $\sum_n p_n\left(\frac{1}{z - a_n}\right)$. Convergence?

$\boxed{3}$ Quick Review of the last lectures in Math 220A

Sequences $\{f_n\}$ holomorphic in $\mathcal{U} \subseteq \mathbb{C}$

Recall that the notion of convergence we considered was

local uniform convergence $\Longleftrightarrow$ convergence on compact subsets

$$f_n \overset{l.u.}{\rightrightarrows} f \qquad \Longleftrightarrow \qquad f_n \overset{c}{\rightrightarrows} f$$

Weierstraß Convergence Theorem

Let $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

$\boxed{1}$ f holomorphic

$\boxed{2}$ $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Series $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic. Assume

$(*)$ $\forall\ K \subseteq \mathcal{U}$ compact $\exists\ M_n(K)$, $|f_n| \leq M_n(K)$ over K. & $\sum_{n=1}^{\infty} M_n(K) < \infty$.

'normal convergence"

M-test

$\Rightarrow f = \sum_{n=1}^{\infty} f_n$ converges absolutely & uniformly on every K.

Weierstraß Thm

$\Rightarrow f$ holomorphic & $f' = \sum_{n=1}^{\infty} f_n'$

This quarter

$\boxed{I}$ infinite products $\prod_{n=1}^{\infty} f_n(z)$ Weierstraß

$\boxed{II}$ series of meromorphic functions $\sum_{n=1}^{\infty} f_n(z)$

Mittag-Leffler

[4] Infinite Products — Conway VII. 5.

Main Question Given $f_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, how do we define $f(z) = \prod_{k=1}^{\infty} f_k(z)$? Furthermore,

[i] Is f holomorphic?

[ii] Is it true that $\text{Zero}(f) = \bigcup_k \text{Zero}(f_k)$?

Step back: Given $p_k \in \mathbb{C}$, how to define

$$P = \prod_{k=1}^{\infty} p_k \ ?$$

Wrong answer Form the partial products

$$P_n = \prod_{k=1}^{n} p_k \quad \text{and define} \quad P = \lim_{n \to \infty} P_n$$

Issues [i] If $p_\ell = 0 \Rightarrow P = 0$ no matter what the other p_n's are. Thus one term would determine convergence of the product which is unfair.

[ii] We could have $P = 0$ even though $p_k \neq 0 \ \forall k$

e.g. $\prod_{k=1}^{\infty} \frac{1}{k} = 0$. Thus we have no control over the zeroes of a product of functions.

Question What kind of products will we consider?

Definition $\prod_{k=1}^{\infty} p_k = P$ converges iff $\exists M$ such that $\lim_{n\to\infty} \prod_{k=M}^{n} p_k$ exists and equals $\hat{P} \neq 0$. We then set

$$P = p_1 \cdots p_{M-1} \hat{P}$$

Remarks $\boxed{i}$ The value of $\mathcal{P}$ is independent of M (check)

$\boxed{ii}$ Only finitely many terms can be zero.

Indeed $\hat{P} \neq 0 \Rightarrow p_k \neq 0$ for $k \geq M$.

$\boxed{iii}$ With this definition, we can control the zeros.

Indeed, $P = 0 \iff p_1 \cdots p_{M-1} \hat{P} = 0 \quad (\hat{P} \neq 0)$

$\iff p_1 = 0$ or $\ldots$ or $p_{M-1} = 0$

$\iff \exists\, k$ with $p_k = 0$.

same behaviour as finite products.

$\boxed{IV}$ Note

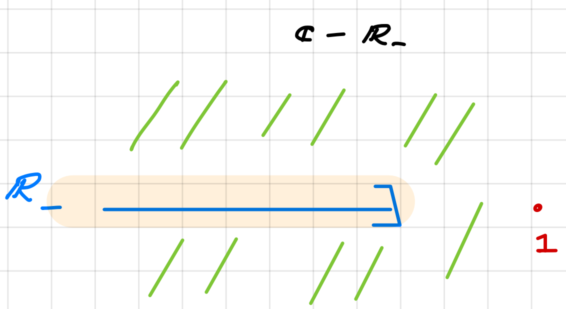
$$p_n = \frac{\prod_{k=N}^{n} p_k}{\prod_{k=N}^{n-1} p_k} \longrightarrow \frac{\hat{p}}{\hat{p}} = 1. \quad \text{as } n \to \infty.$$

Henceforth, we will assume $p_k = 1 + a_k$, $a_k \longrightarrow 0$.

$$\prod_{k=1}^{\infty} (1 + a_k)$$

We seek to connect infinite products to infinite series.

Recall Principal branch of logarithm $z \neq 0$, $z \in \mathbb{C} \setminus \mathbb{R}_-$



$$\operatorname{Log}(z) = \log r + i\theta$$

$$\theta \in (-\pi, \pi)$$

$\operatorname{Log}(1+z)$. makes sense for z small since $1 + z \notin \mathbb{R}_-$.

Lemma $\prod_{k=1}^{\infty} (1+a_k)$ converges $\iff$ $\exists\ N>0$ such that $\sum_{k=N}^{\infty} \operatorname{Log}(1+a_k)$ converges

Conway V. 5. 2

Proof Write

$$S_n = \sum_{k=N}^{n} \operatorname{Log}(1+a_k) \qquad \Longrightarrow \qquad e^{S_n} = P_n.$$

$$P_n = \prod_{k=N}^{n} (1+a_k)$$

„$\Leftarrow$": If $S_n \to S$, $P_n = e^{S_n} \longrightarrow e^{S} = \widehat{P} \neq 0$.

„$\Rightarrow$": Assume $P_n \to \widehat{P}$. We wish to show $S_n \to S$.

Pick α such that $\widehat{P} \notin \mathbb{R}_{\geq 0} e^{i\alpha}$. We use the branch $\operatorname{Log}_\alpha$

$$\operatorname{Log}_\alpha z = \log r + i\theta, \quad \theta \in (\alpha, \alpha+2\pi)$$

$$e^{S_n} = P_n \implies S_n = \operatorname{Log}_\alpha P_n + 2\pi i \, \ell_n, \quad \ell_n \in \mathbb{Z}.$$

$$\Vert$$

$$e^{\operatorname{Log}_\alpha P_n}$$

We claim $\ell_n = \ell_{n-1}$ if $n >> 0$ $\Rightarrow \exists\, \ell,\ \ell_n = \ell$.

$$\Rightarrow\ s_n = \mathrm{Log}_\alpha\, p_n + 2\pi i\, \ell_n \longrightarrow \mathrm{Log}_\alpha\, \hat{p} + 2\pi i \ell := s$$

continuity of Log_α

To prove the claim, consider away from $\mathbb{R}_{\geq 0} e^{i\alpha}$.

$$\underbrace{s_n - s_{n-1}}_{\downarrow\, 0} = \underbrace{\mathrm{Log}_\alpha\, p_n - \mathrm{Log}_\alpha\, p_{n-1}}_{\downarrow\, 0} + 2\pi i\,(\ell_n - \ell_{n-1})$$

as $n \to \infty$

Note $s_n - s_{n-1} = \mathrm{Log}\,(1 + a_n) \longrightarrow \mathrm{Log}\, 1 = 0$

$$\mathrm{Log}_\alpha\, p_n - \mathrm{Log}_\alpha\, p_{n-1} \longrightarrow \mathrm{Log}_\alpha\, \hat{p} - \mathrm{Log}_\alpha\, \hat{p} = 0$$

This shows $\ell_n - \ell_{n-1} \longrightarrow 0$ as $n \to \infty$

$\ell_n - \ell_{n-1} \in \mathbb{Z}$

$\Rightarrow \ell_n = \ell_{n-1}$ if $n >> 0$.

Absolute convergence

Question How do we define absolutely convergent products $\prod_{k=1}^{\infty} p_k$

Wrong Answer: $\prod_{k=1}^{\infty} |p_k|$ converges

But then for $p_k = (-1)^k$, $\prod_{k=1}^{\infty} (-1)^k$ converges absolutely which is absurd.

Def $\prod_{k=1}^{\infty} (1+a_k)$ converges absolutely iff $\exists N$ such that

$\sum_{k=N}^{\infty} \operatorname{Log}(1+a_k)$ converges absolutely.

Lemma TFAE Conway V.5.4/V.5.6

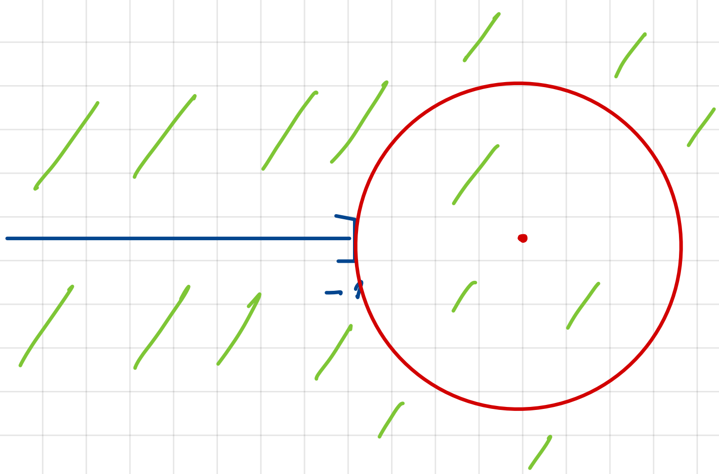
[i] $\prod_{k=1}^{\infty}(1+a_k)$ converges absolutely

[ii] $\sum_{k=1}^{\infty} a_k$ converges absolutely

[iii] $\prod_{k=1}^{\infty}(1+|a_k|)$ converges

Proof Consider Taylor expansion in $\Delta(0,1) \subseteq \mathbb{C} \setminus (-\infty,-1]$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$



$$\Downarrow$$

$$\frac{\operatorname{Log}(1+z)}{z} = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots$$

$\Rightarrow \lim_{z\to 0} \frac{\operatorname{Log}(1+z)}{z} = 1 \Rightarrow \exists\, \rho > 0$ such that if $|z|<\rho,\ z\neq 0$.

$$\frac{1}{2} \le \left|\frac{\operatorname{Log}(1+z)}{z}\right| \le \frac{3}{2}$$

Important inequality $\exists\, \rho$ s.t. if $|z|<\rho$

$$\frac{1}{2}|z| \le |\operatorname{Log}(1+z)| \le \frac{3}{2}|z|.$$

$\boxed{i} \iff \boxed{ii}$ By defn, $\prod_{k=1}^{\infty}(1+a_k)$ converges absolutely

$\iff \sum_{k=N}^{\infty} \mathrm{Log}(1+a_k)$ converges absolutely

$\iff \sum_{k=N}^{\infty} a_k$ converges absolutely (comparison test + important inequality)

Finally,

$\boxed{ii} \iff \sum_{k=N}^{\infty} |a_k|$ converges absolutely

$\iff \prod_{k=1}^{\infty}(1+|a_k|)$ converges absolutely by $\boxed{i} \iff \boxed{ii}$ for $\tilde{a}_k = |a_k|$

$\iff \prod_{k=1}^{\infty}(1+|a_k|)$ converges $\iff \boxed{iii}$

indeed, absolute convergence of the product is superfluous

$$\sum_{k=N}^{\infty} |\mathrm{Log}(1+|a_k|)| = \sum_{k=N}^{\infty} \mathrm{Log}(1+|a_k|)$$

Remark (Rearrangements).

Math 140A we learned that if $\sum_{k=1}^{\infty} b_k$ is

absolutely convergent then $\forall\ \sigma : \mathbb{N} \longrightarrow \mathbb{N}$ bijection

then $\sum_{k=1}^{\infty} b_{\sigma(k)}$ converges to the same sum.

The same happens for absolutely convergent products

$\prod_{k=1}^{\infty} p_k$ can be rearranged.

Indeed, take $b_k = \operatorname{Log}(1+a_k)$, $p_k = 1+a_k$.