Math 220B — Lecture 10

February 14, 2024

Last time

$\Delta = \Delta(0,1)$, $f : \Delta \longrightarrow \Delta$

- if $f(0) = 0$ then
  - we proved Schwarz Lemma
  - we determined $f \in \mathrm{Aut}\,\Delta$, $f(0) = 0$
- if $f(0) \neq 0$
  - we determined $f \in \mathrm{Aut}\,\Delta$

Idea Use φ_a to recenter f so that 0 maps to 0.

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a} z}.$$

Application – Fixed points (Often on Qualifying Exam)

Show if $f : \Delta \longrightarrow \Delta$ holomorphic, $f \neq \mathbb{1} \Rightarrow f$ has at most 1 fixed point.

Proof Assume $f(a) = a$ & $f(b) = b$. & $a \neq b$.

If $a = 0$ then $f(0) = 0$ & $f(b) = b \Longrightarrow f$ rotation via Schwarz $f(z) = e^{i\theta} z$. Using $f(b) = b \Rightarrow e^{i\theta} = 1 \Rightarrow$

$\Rightarrow f = \mathbb{1}$ which is disallowed.

For $a \neq 0$, we reduce to this case. Let

$$\tilde{f} = \varphi_a f \varphi_a^{-1} \quad \& \quad \lambda = \varphi_a(b) \neq 0 = \varphi_a(a).$$

$$\begin{tikzcd}
a \arrow[r, bend left, "\varphi_a"] \arrow[d, "f"'] & 0 \arrow[l, bend left, "\varphi_a^{-1}"] \arrow[d, "\tilde{f}"] \\
a \arrow[r, "\varphi_a"'] & 0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
b \arrow[r, bend left, "\varphi_a"] \arrow[d, "f"'] & \lambda \arrow[l, bend left, "\varphi_a^{-1}"] \arrow[d, "\tilde{f}"] \\
b \arrow[r, "\varphi_a"'] & \lambda
\end{tikzcd}$$

Then $\tilde{f}(0) = 0$ and $\tilde{f}(\lambda) = \lambda \Rightarrow \tilde{f} = \mathbb{1} \Rightarrow$

$\Rightarrow \varphi_a f \varphi_a^{-1} = \mathbb{1} \Rightarrow f = \mathbb{1}$. again a contradiction.

Thus f has at most one fixed point.

1. Schwarz – Pick

Conway VI. 2. 3.

Question Is there a version of Schwarz if $f(0) \neq 0$?

Yes — Schwarz – Pick Lemma.

— we illustrate it for derivatives

Proposition $f : \Delta \longrightarrow \Delta$ holomorphic, $\forall a \in \Delta$

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

Remark If $a = 0$ this gives $|f'(0)| \leq 1 - |f(0)|^2$.

If $f(0) = 0$ this gives $|f'(0)| \leq 1$. Thus the Proposition generalizes Schwarz Lemma.

Remark This is naturally formulated in hyperbolic geometry.

Proof We know this when $a = 0$ & $\alpha = f(a) = 0$.

We use $\mathrm{Aut}(\Delta)$ to reduce to this case.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, shift right, "\varphi_a"'] & \alpha \arrow[d, "\varphi_\alpha"] \\ 0 \arrow[r, "\tilde f"'] \arrow[u, shift right, "\varphi_{-a}"'] & 0 \end{tikzcd}$$

Let $f(a) = \alpha$. Let

$$\tilde f = \varphi_\alpha \circ f \circ \varphi_{-a} \Rightarrow \tilde f(0) = 0$$

as the diagram shows.

By Schwarz, $|\tilde f'(0)| \leq 1$. We compute using the chain rule

$$\tilde f'(0) = \varphi_\alpha'(f(\varphi_{-a}(0))) \cdot f'(\varphi_{-a}(0)) \cdot \varphi_{-a}'(0)$$

$$= \varphi_\alpha'(\alpha) \cdot f'(a) \cdot \varphi_{-a}'(0)$$

$$= \frac{1}{1-|\alpha|^2} \cdot f'(a) \cdot (1-|a|^2)$$

& $|\tilde f'(0)| \leq 1$ gives

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}$$

as needed.

Example Conway VI.2.3 $f: \Delta \longrightarrow \Delta$ holomorphic.

If $f\left(\frac{1}{2}\right) = \frac{1}{4}$, find the maximum value of $\left|f'\left(\frac{1}{2}\right)\right|$.

Remark

Schwarz $f(0)=0$		Schwarz – Pick
$\lvert f'(0)\rvert \leq 1$	$\Longleftarrow$	$\lvert f'(a)\rvert \leq \dfrac{1-\lvert f(a)\rvert^2}{1-\lvert a\rvert^2}$
$\lvert f(z)\rvert \leq \lvert z\rvert$		?

Define $d(z,w) = \left|\dfrac{z-w}{1-\bar{z}w}\right|$ = pseudo hyperbolic distance

Schwarz – Pick Holomorphic maps decrease pseudo hyperbolic distance.

This will be made precise in HWK 4

2. Further applications of Schwarz

We can use Schwarz to study other domains e.g.

$\boxed{1}$ $\mathcal{U} = \Delta^{\times} = \Delta(0,1) \setminus \{0\}$

$\boxed{11}$ $\mathcal{U} = \mathfrak{H}^{+}$ = upper half plane

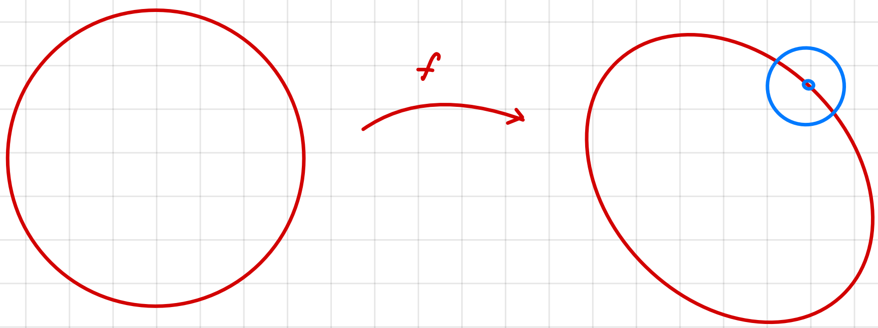
$\boxed{1}$ Example All automorphisms of $\Delta^{\times}$ are rotations.

Proof Let $f: \Delta^{\times} \longrightarrow \Delta^{\times}$. Since $\operatorname{Im} f$ is bounded $\Rightarrow$

$\Rightarrow f$ can be extended across 0 by the removable singularity theorem. The extension $\tilde{f}: \Delta \longrightarrow \overline{\Delta}$ is holomorphic.

Its image $\operatorname{Im} \tilde{f} \subseteq \Delta$ by the open mapping theorem

(draw picture)



We claim $\tilde{f}(0) = 0$. Then $f: \Delta^\times \longrightarrow \Delta^\times$ shows $\tilde{f}$ bijective from $\Delta \longrightarrow \Delta$ hence a biholomorphism preserving 0. Then $\tilde{f}$ is a rotation. (by last time)

To show $\tilde{f}(0) = 0$ assume otherwise $\tilde{f}(0) = \alpha \neq 0$.

Since $\alpha \in \Delta^\times$ we can find $a \in \Delta^\times$, $f(a) = \alpha$.

By the open mapping theorem, we can find small discs

Δ_0, Δ_a, Δ_α near $0, a, \alpha$ with $\Delta_0 \cap \Delta_a = \emptyset$ and.

$$\Delta_\alpha \subseteq \tilde{f}(\Delta_0), \quad \Delta_\alpha \subseteq f(\Delta_a). \quad (\text{why?}).$$

Let $b \in \Delta_\alpha \setminus \{\alpha\} \Rightarrow b \in \tilde{f}(\Delta_0) \Rightarrow b = f(u),\ u \neq 0,\ u \in \Delta_0$

$$\Rightarrow b \in f(\Delta_a) \Rightarrow b = f(v),\ v \in \Delta_a$$

$\Rightarrow f(u) = f(v) = b$

$u \neq v$ since $\Delta_0 \cap \Delta_a = \emptyset$

$\Rightarrow f$ not injective (contradiction).

$\boxed{\text{II}}$ Upper half plane

Key idea: Use $\mathfrak{H}^+ \xrightarrow{c} \Delta$, $\quad c(z) = \dfrac{z-i}{z+i}$

$$c^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

Questions we can answer:

$\boxed{\text{I}}$ $\mathrm{Aut}(\mathfrak{H}^+)$

Schwarz lemma for $f: \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$

Schwarz-Pick for $f: \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$.

$\boxed{\text{II}}$ Biholomorphisms $\Delta \longrightarrow \mathfrak{H}^+$

Schwarz Lemma for $f: \Delta \longrightarrow \mathfrak{H}^+$

Schwarz-Pick for $f: \mathfrak{H}^+ \longrightarrow \Delta$

for derivatives or for distance ...

It is impossible to record them all.

Example $f: \Delta \to \mathbb{C}$, $\operatorname{Im} f(z) > 0 \ \forall z \in \Delta$, $f(0) = i$. Show

$$|f'(0)| \leq 2.$$

Note $f: \Delta \to \mathfrak{h}^+$.

Let $\tilde{f} = c \circ f$. Then $\tilde{f}(0) = 0$ since $c(i) = \left.\frac{z-i}{z+i}\right|_{z=i} = 0$

$\Rightarrow |\tilde{f}'(0)| \leq 1$ by Schwarz. We compute

$$|\tilde{f}'(0)| = |c'(f(0)) \cdot f'(0)| = |c'(i) \cdot f'(0)| \leq 1.$$

Since $c'(i) = \frac{1}{2i}$ $\Rightarrow |f'(0)| \leq 2.$

8. Further discussion of Aut. — Loose ends

[i] Aut $\mathbb{C}$ $= \{az+b : a \neq 0,\ b \in \mathbb{C}\} \cong \text{Aff}.$

[ii] Aut $\hat{\mathbb{C}}$ $= PGL_2(\mathbb{C})$

[iii] Aut Δ $\cong SU(1,1)/_{\pm 1} = PSU(1,1)$

[iv] Aut $\mathfrak{H}^+$ $= SL(2,\mathbb{R})/_{\pm 1.} = PSL(2,\mathbb{R})$

[v] Aut $\Delta^\times$ $\cong$ Rotations

Case [i] $\quad \mathcal{U} = \mathbb{C}$

7. Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Show that $f(z) = az + b$. You can solve this problem using the notions introduced in Problem 6 above.

Math 220 A, Homework 5.