Math 220B — Lecture 11

February 21, 2024

1. Further discussion of Aut.

✓ $\boxed{i}$ $\operatorname{Aut} \mathbb{C} = \{az+b : a \neq 0,\ b \in \mathbb{C}\} \cong \mathrm{Aff}.$

⇝ Math 220 A

✓ $\boxed{ii}$ $\operatorname{Aut} \Delta^{\times} \cong$ Rotations

⇝ last time

✓ $\boxed{iii}$ $\operatorname{Aut} (\Delta \setminus \{0, a\}) \cong \mathbb{Z}/2\mathbb{Z}$

⇝ homework

Today

$\boxed{iv}$ $\operatorname{Aut} \widehat{\mathbb{C}} = PGL_2(\mathbb{C})$

$\boxed{v}$ $\operatorname{Aut} \Delta \cong SU(1,1)/\pm 1 = PSU(1,1)$

$\boxed{vi}$ $\operatorname{Aut} \mathfrak{H}^{+} = SL(2,\mathbb{R})/\pm 1. = PSL(2,\mathbb{R})$

Case [iv] $\quad \mathcal{U} = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$

Mobius transforms - Math 220A, Lecture 3.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies h_M(z) = \frac{az+b}{cz+d}, \quad h_M : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$h_M = h_N \iff M = \lambda N.$$

$$h_M \, h_N = h_{MN}.$$

h_M bijective $\iff$ M invertible since $h_M \circ h_{M^{-1}} = \mathbb{I}$

Define $PGL_2 = GL_2 \Big/ \{\lambda \cdot \mathbb{I},\ \lambda \neq 0\}$ = invertible 2×2 complex matrices up to scaling.

Recall from Math 220 A, Lecture 3, the action of Mobius transforms is transitively on $\hat{\mathbb{C}}$.

Theorem $\mathrm{Aut}\,\hat{\mathbb{C}} = PGL_2$.

Proof If $f \in \mathrm{Aut}\,\hat{\mathbb{C}}$, $f(\infty) = \infty$ then $f : \mathbb{C} \longrightarrow \mathbb{C}$ is bijective. Thus $f(z) = az + b = f_M$ for the matrix

$$M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

If $f(\infty) \neq \infty$ then $f(\infty) = \lambda \in \mathbb{C}$. Let

$$g(z) = \frac{1}{f(z) - \lambda} \Rightarrow g(\infty) = \infty \Rightarrow g(z) = az + b$$

$$\Rightarrow f(z) = \lambda + \frac{1}{az + b} = \text{fractional linear transformation,}$$

as needed.

Case $\boxed{v}$ Aut (Δ).

Question What is Aut (Δ) as an abstract group?

$f \in \mathrm{Aut}\, \Delta$

$$f(z) = e^{i\theta} \cdot \frac{z - a}{1 - \bar{a} z} = \frac{e^{i\theta/2}}{e^{-i\theta/2}} \cdot \frac{z - a}{1 - \bar{a} z} = h_M.$$

$$M = \begin{bmatrix} e^{i\theta/2} & -a e^{i\theta/2} \\ -\bar{a} e^{-i\theta/2} & e^{-i\theta/2} \end{bmatrix} = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \text{ invertible.}$$

Note $\det M = 1 - |a|^2 > 0$. Let $\lambda = (1 - |a|^2)^{-1/2}$.

Rescale $A \longrightarrow \lambda A, \quad \lambda \in \mathbb{R}.$

$$\Longrightarrow \quad A\bar{A} - B\bar{B} = |A|^2 - |B|^2 = 1.$$

$B \longrightarrow \lambda B, \quad \lambda \in \mathbb{R}.$

Conclusion
$$\mathrm{Aut}\, \Delta = \left\{ \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} : |A|^2 - |B|^2 = 1 \right\} \Big/ \pm 1$$

$$= SU(1,1) / \pm 1 = PSU(1,1).$$

Case VII   Aut $\mathfrak{H}^+$

Key idea Use Cayley transform:

$$\begin{tikzcd} \mathfrak{H}^+ \arrow[r, shift left, "C"] & \Delta \arrow[l, shift left, "C^{-1}"] \end{tikzcd}$$

$$C(z) = \frac{z-i}{z+i}$$

$$C^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

$$\begin{tikzcd} \mathfrak{H}^+ \arrow[r, "C", "\sim"'] \arrow[rrr, bend right, "g"', "\sim"] & \Delta \arrow[r, "f", "\sim"'] & \Delta \arrow[r, "C^{-1}", "\sim"'] & \mathfrak{H}^+ \end{tikzcd}$$

$g = C^{-1} \circ f \circ C$ is an automorphism

Any $g \in \operatorname{Aut} \mathfrak{H}^+$ is of this form for $f = C g C^{-1}$.

$$\text{Compute}\quad \overbrace{C^{-1}}^{\frac{1}{2}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}} \underbrace{\begin{bmatrix} A & B \\ \overline{B} & \overline{A}\end{bmatrix}}_{\text{Aut}\,\Delta} \overbrace{C}^{\begin{bmatrix}1 & -i\\ 1 & +i\end{bmatrix}} = \begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}$$

$$\alpha = \operatorname{Re} A + \operatorname{Re} B \qquad \delta = \operatorname{Re} A - \operatorname{Re} B$$

$$\beta = \operatorname{Im} A - \operatorname{Im} B \qquad \gamma = -\operatorname{Im} A - \operatorname{Im} B$$

$$\Longrightarrow \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

$$|A|^2 - |B|^2 = 1 \quad \Longleftrightarrow \quad (\operatorname{Re} A)^2 + (\operatorname{Im} A)^2 - (\operatorname{Re} B)^2 - (\operatorname{Im} B)^2 = 1.$$

$$\Longleftrightarrow \alpha\delta - \beta\gamma = 1.$$

$$\Rightarrow \begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix} \in SL(2, \mathbb{R}).$$

Conclusion

$$\text{Aut}(\mathfrak{h}^+) \cong SL(2, \mathbb{R}) / \{\pm 1\} = PSL(2, \mathbb{R}).$$

II Riemann Mapping theorem

GRUNDLAGEN

FÜR EINE

ALLGEMEINE THEORIE DER FUNCTIONEN

EINER

VERÄNDERLICHEN COMPLEXEN GRÖSSE.

VON

(Georg Friedrich) Bernhard
B. RIEMANN.

ZWEITER, UNVERÄNDERTER ABDRUCK.

GÖTTINGEN,
VERLAG VON ADALBERT RENTE.
1867.

Riemann's thesis was published in 1851.

"Two given simply connected planar surfaces can always be related to each other in such a way that every point of one corresponds to one point of another, which varies continuously with it, and their corresponding smaller parts are similar."

(Translation by R. Remmert).

Theorem $\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow$ $\mathcal{U}$ biholomorphic to the unit disc. $\Delta = \Delta(0,1)$.

Remarks $\boxed{I}$ $\mathcal{U} = \mathbb{C}$ is not biholomorphic to Δ.

By Liouville, there cannot exist a holomorphic nonconstant map $\mathbb{C} \longrightarrow \Delta$.

$\boxed{II}$ Implications in topology

$\mathcal{U}$ simply connected, $\mathcal{U} \subseteq \mathbb{C}$. $\Rightarrow$ $\mathcal{U}$ is topologically Δ i.e.

$\exists$ bicontinuous map $\mathcal{U} \longrightarrow \Delta$ (homeomorphism).

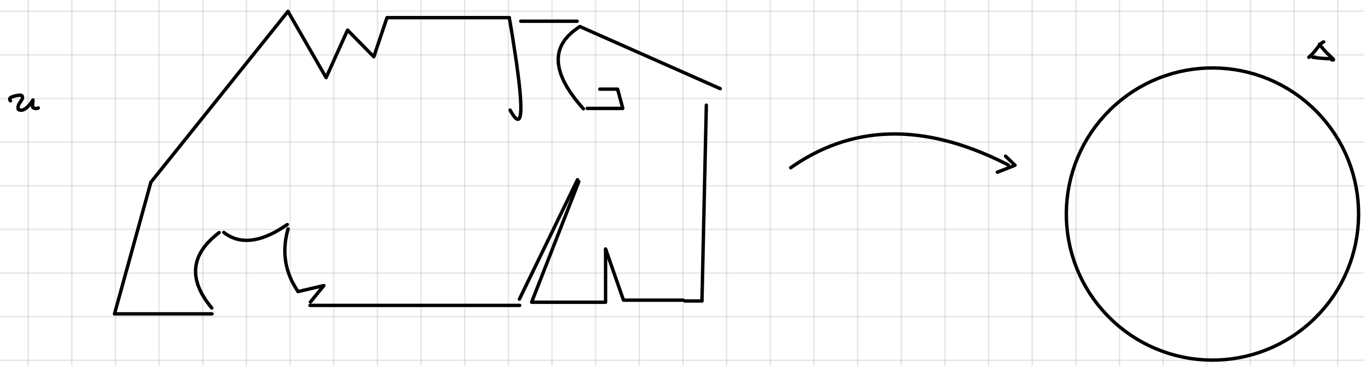
This holds even for $\mathcal{U} = \mathbb{C}$ using the map:

not holomorphic.

$$\mathbb{C} \longrightarrow \Delta, \quad z \longrightarrow \frac{z}{\sqrt{1+|z|^2}}$$

Why is the proof difficult?

Imagine the domain



It is hard to construct explicit maps (even in the topological category).

Examples

[i] $c : \mathfrak{H}^+ \longrightarrow \Delta$, $\quad c(z) = \dfrac{z-i}{z+i}$.

[ii] biholomorphism between Δ and the slit plane $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (both simply connected).

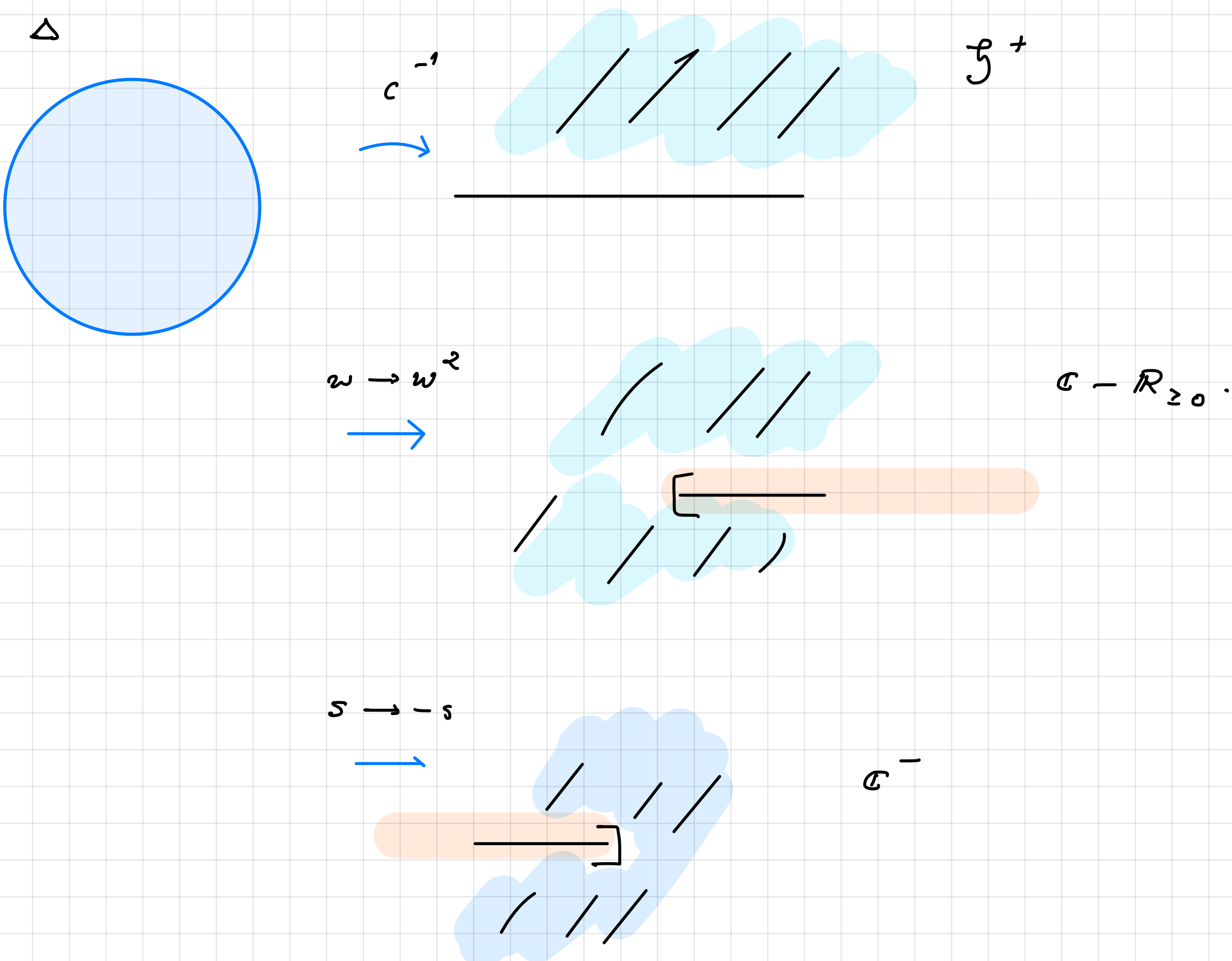
We use simple geometric moves:

$$\Delta \longrightarrow \mathfrak{h}^+ \quad \text{via} \quad c^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

$$\mathfrak{h}^+ \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\geq 0} \quad \text{via} \quad w \longrightarrow w^2.$$

$$\mathbb{C} \setminus \mathbb{R}_{\geq 0} \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\leq 0} = \mathbb{C}^- \quad \text{via} \quad s \longrightarrow -s.$$

Composition: $-\left(i \cdot \frac{1+z}{1-z}\right)^2 = \left(\frac{1+z}{1-z}\right)^2 : \Delta \longrightarrow \mathbb{C}^-$.



Riemann Mapping theorem

Theorem $\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow$ $\mathcal{U}$ biholomorphic to the unit disc. $\Delta = \Delta(0,1)$.

Ingredients in the proof

[i] Montel & normal families

[ii] Hurwitz's Theorem

[iii] Aut Δ & Schwarz Lemma

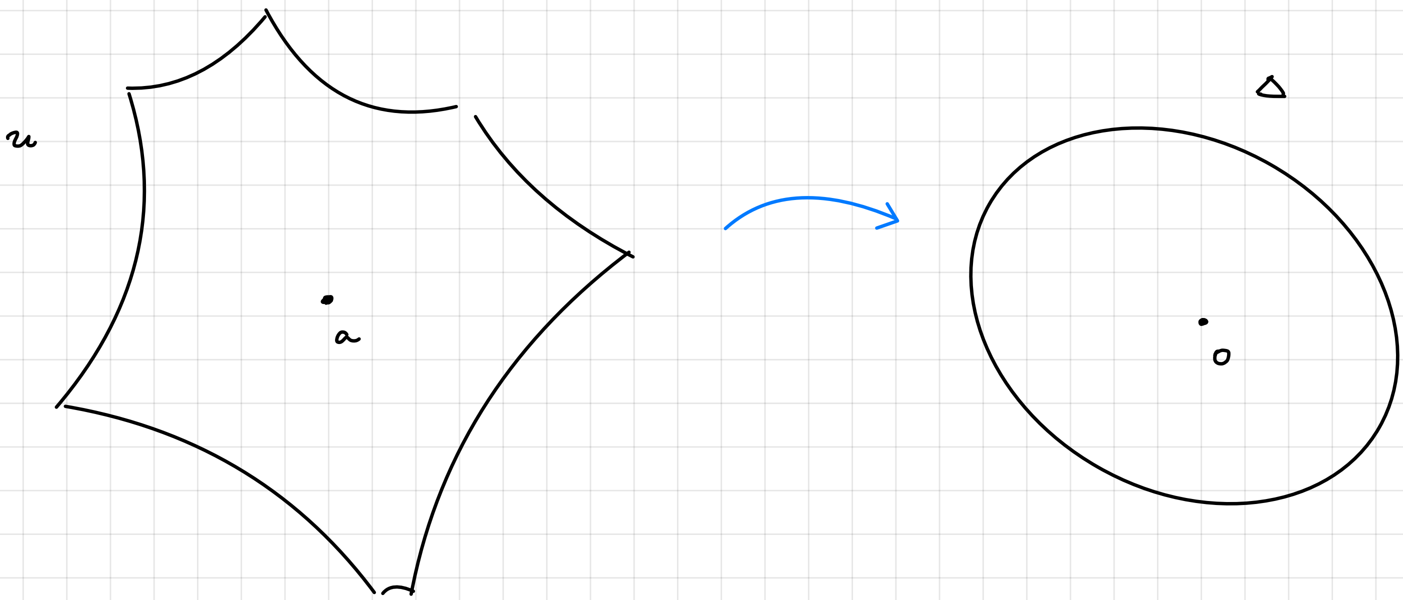
[iv] Square root trick of Carathéodory - Koebe.

& standard tools:

[v] Open Mapping & Weierstraß. convergence

We had to wait to develop these tools.

Strategy Fix $a \in \mathcal{U}$



Want $f : \mathcal{U} \longrightarrow \Delta$ & $f(a) = 0$ & f bijective.

Goal #1 First, $f : \mathcal{U} \longrightarrow \Delta$, $f(a) = 0$, &

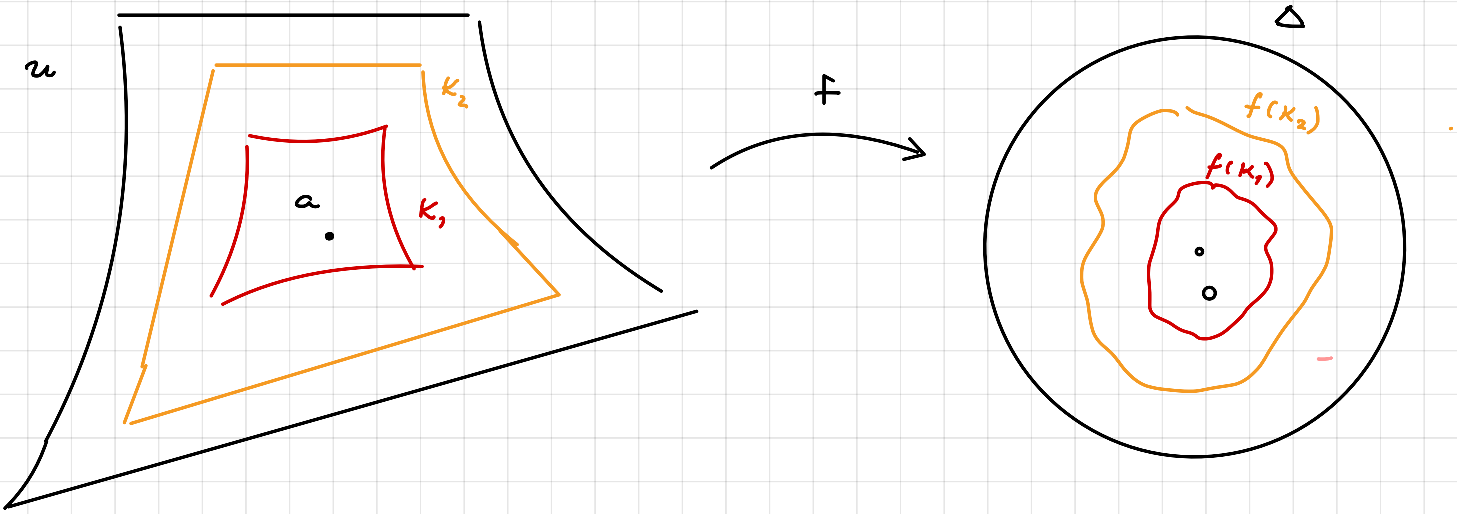
f injective

Main Actor in the Proof Consider the family

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta, \ f(a) = 0, \ f \text{ injective} \}.$$

Want $\mathcal{F} \neq \emptyset$.

Question How to achieve f bijective?



Imagine $\mathcal{U} = \bigcup_n K_n$, $a \in K_n \subseteq \operatorname{Int} K_{n+1}$.

We hope $\bigcup_n f(K_n)$ cover Δ. We expect that this has a chance if $|f'(a)|$ is as large as possible.

Let $M = \sup\{|f'(a)| : f \in \mathcal{F}\}$.

Goal #2 Show $\exists f \in \mathcal{F}$ with $|f'(a)| = M$.

Goal #3 Show that for this choice, $f : \mathcal{U} \longrightarrow \Delta$ is bijective

Why might this actually work?

Example $\mathcal{U} = \Delta$, $a = 0$.

$$\mathcal{F} = \{ f : \Delta \longrightarrow \Delta,\ f(0) = 0,\ f \text{ injective} \}.$$

By Schwarz Lemma, $|f'(0)| \leq 1$. If the maximum value $|f'(0)| = 1$ then f is a rotation so f is bijective.

Remark

We can also consider points $a \in \Delta$, $a \neq 0$. Let

$$\mathcal{F} = \{ f : \Delta \longrightarrow \Delta,\ f(a) = 0,\ f \text{ injective} \}.$$

Schwarz - Pick

$$|f'(a)| \leq \frac{1}{1 - |a|^2}$$ with equality iff

$$f = \mathrm{Rot} \circ \varphi_a \Rightarrow f \text{ bijective.}$$

Question How do we use U simply connected?

Answer Math 220A, Homework 4

2. Assume $f : U \to \mathbb{C}$ is a holomorphic function on a simply connected open set U such that $f(z) \neq 0$ for all $z \in U$. Let $n \geq 2$ be an integer. Show that there is a holomorphic function $g : U \to \mathbb{C}$ such that

$$g(z)^n = f(z).$$

Hint: This has something to do with problem 1(ii).

We only need $n = 2$.

U simply connected $\Rightarrow$ any $f : U \longrightarrow \mathbb{C}$ holomorphic, nowhere zero, admits a holomorphic square root $g : U \to \mathbb{C}$

$$f = g^2 \quad (*)$$

"Root domain"

- $U \subseteq \mathbb{C}$ is a root domain if $(*)$ is satisfied.

Remark simply connected $\Rightarrow$ root domain

Remark This turns out to be equivalent to $\mathcal{U}$ simply connected

We will prove: the seemingly stronger form:

Riemann Mapping Theorem

$\mathcal{U} \neq \mathbb{C}$ root domain $\Rightarrow$ $\mathcal{U}$ is biholomorphic to Δ.