Math 220B — Lecture 12

February 26, 2024

Last time

220A, HWK4

$\mathcal{U}$ simply connected $\Rightarrow$ any $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, no where zero, admits a holomorphic square root $g: \mathcal{U} \to \mathbb{C}$

$$f = g^2 \quad (*)$$

"Root domain"

· $\mathcal{U} \subseteq \mathbb{C}$ is a root domain if $(*)$ is satisfied.

simply connected $\Rightarrow$ root domain

Riemann Mapping Theorem

$\mathcal{U} \neq \mathbb{C}$ root domain $\Rightarrow$ $\mathcal{U}$ is biholomorphic to Δ.

Today

— execute the strategy outlined last time.

Proof of Riemann Mapping

Fix $a \in \mathcal{U}$. Let

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta : f \text{ holomorphic, injective, } f(a) = 0 \}.$$

Step 1 If $\mathcal{U}$ is a root domain, $\mathcal{U} \neq \mathbb{C} \implies \mathcal{F} \neq \emptyset$

Proof Let $b \notin \mathcal{U}$, which is possible since $\mathcal{U} \neq \mathbb{C}$.

Consider $h(z) = z - b$, $h : \mathcal{U} \longrightarrow \mathbb{C}$. Note $h(z) \neq 0$ for $z \in \mathcal{U}$, since $b \notin \mathcal{U}$. Thus h admits a square root

$g : \mathcal{U} \longrightarrow \mathbb{C}$, $g(z)^2 = z - b$.

Claim 1 g injective.

Indeed, if $g(z_1) = g(z_2) \implies g(z_1)^2 = g(z_2)^2 \implies$

$\implies z_1 - b = z_2 - b \implies z_1 = z_2$.

Claim 2 $g(\mathcal{U}) \cap (-g)(\mathcal{U}) = \emptyset$.

Indeed, if $\exists\, z_1, z_2 \in \mathcal{U}$ with $g(z_1) = -g(z_2)$

$\Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow z_1 - b = z_2 - b \Rightarrow z_1 = z_2$.

But then $g(z_1) = -g(z_2) \Rightarrow g(z_1) = -g(z_1) \Rightarrow g(z_1) = 0$

$\Rightarrow g(z_1)^2 = 0 = z_1 - b \Rightarrow z_1 = b$. But $z_1 \in \mathcal{U}$, $b \notin \mathcal{U}$.

Claim 3 $\exists\, c, r$ with $|g(z) - c| > r \quad \forall z \in \mathcal{U}$.

Indeed, by the open mapping theorem, $(-g)(\mathcal{U})$ is open so it contains a disc $\overline{\Delta}(c,r)$. By Claim 2,

$g(\mathcal{U}) \subseteq \mathbb{C} \setminus \overline{\Delta}(c,r) \iff |g(z) - c| > r \quad \forall z \in \mathcal{U}$.

Construction Let $f(z) = \dfrac{r}{g(z) - c}$. $\Rightarrow f$ injective since g is by Claim 1 & $f : \mathcal{U} \longrightarrow \Delta(0,1)$. by Claim 3.

To achive $f(a) = 0$, define $\tilde{f}(z) = \dfrac{f(z) - f(a)}{2}$.

$\Rightarrow \tilde{f}$ injective since f is. & $\tilde{f}(a) = 0$.

Note that since f takes values in Δ, the same is true for $\tilde{f}$

$$|\tilde{f}(z)| \leq \frac{1}{2}\left(|f(z)| + |f(a)|\right) < \frac{1}{2}(1+1) = 1$$

Thus $\tilde{f} \in \mathcal{F} \Rightarrow \mathcal{F} \neq \Phi$.

Step 2 Let $M = \sup\{|f'(a)|, f \in \mathcal{F}\}$

Show: The supremum is achieved by some $f \in \mathcal{F}$.

Proof: Indeed, take $f_n \in \mathcal{F}$ with $|f_n'(a)| \to M$ as $n \to \infty$

The family $\mathcal{F}$ is bounded by 1 since the functions in $\mathcal{F}$ take values in Δ. $\xRightarrow{\text{Montel}}$ $\mathcal{F}$ normal. $\Rightarrow$

$\Rightarrow$ passing to a subsequence, we may assume

$f_n \rightrightarrows f$ locally uniformly.

Claim 4 f holomorphic, $f(a) = 0$, $|f'(a)| = M$.

Indeed, by Weierstraß convergence, f is holomorphic.

and $f_n' \rightrightarrows f'$ locally uniformly. In particular,

$f_n'(a) \longrightarrow f'(a)$ so $|f'(a)| = M$.

Since $f_n(a) = 0$ & $f_n \rightrightarrows f$ at a, we have

$f(a) = 0$.

Claim 5. $f : U \longrightarrow \Delta$ & f injective.

Indeed, f_n injective & $f_n \overset{l.u.}{\rightrightarrows} f$ shows f is either

injective or f constant by Hurwitz's theorem

(Math 220 A, Lecture 19).

If f = constant $\Rightarrow f'(a) = 0 \Rightarrow M = 0 \Rightarrow$

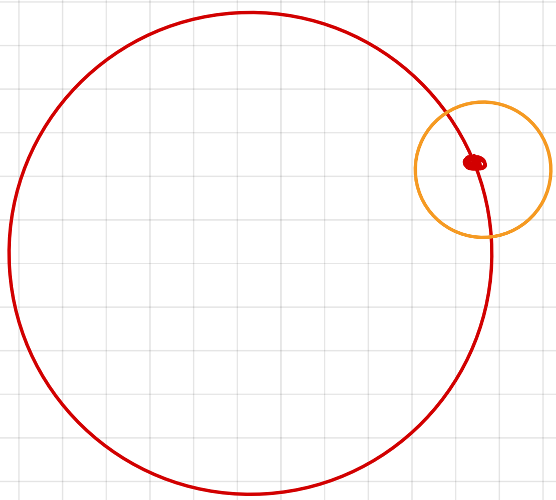
$\Rightarrow g'(a) = 0 \quad \forall\, g \in \mathcal{F}$ since M is the supremum.

But if $g \in \mathcal{F}$, g injective and $g'(a) \neq 0$.

Indeed, let $V = \mathrm{Im}\, g$, so $g: U \longrightarrow V$ is bijective hence a biholomorphism. Let $h: V \to U$ be the inverse. Then $h \circ g = \mathbb{1} \Rightarrow h'(g(a)) \cdot g'(a) = 1$ by chain rule $\Rightarrow g'(a) \neq 0$.

Thus f injective.

Note that since $f_n \overset{l.u}{\rightrightarrows} f$ and $f_n: U \longrightarrow \Delta$ shows $f: U \longrightarrow \overline{\Delta}$. By the open mapping theorem, $f: U \longrightarrow \Delta$ ($f \neq$ not constant).



By Claims 4 & 5, $f \in \mathcal{F}$ and $|f'(a)| = M \Rightarrow$ Step 2 ✓.

Step 3

For the extremal function f in Step 2

we show f surjective. then f biholomorphism

If $f: U \longrightarrow \Delta$ not surjective then we show $\exists\, \tilde{f} \in \mathcal{F}$

with $|\tilde{f}'(a)| > |f'(a)|$ contradicting maximality of $|f'(a)|$.

Strategy

We will in fact show that if $f: U \longrightarrow \Delta$ not surjective

then $\exists\, \tilde{f}: U \longrightarrow \Delta$, $F: \Delta \longrightarrow \Delta$, $f = F \circ \tilde{f}$ and

$\tilde{f} \in \mathcal{F}$, $F(0) = 0$, $F \notin \operatorname{Aut} \Delta$.

Assume this can be done. The proof is then completed.

Indeed, by Schwarz Lemma $\Rightarrow$ $|F'(0)| < 1$. (The inequality is strict since F is not a rotation as $F \notin \operatorname{Aut} \Delta$).

Then we indeed contradict maximality since

$$|f'(a)| = |F'(0)| \cdot |\tilde{f}'(a)| < |\tilde{f}'(a)|.$$

How do we execute the above Strategy?

Assume $f : U \longrightarrow \Delta$ is not surjective.

Let $\alpha \in \Delta \setminus f(U)$.

<u>Construction of the function $\tilde{f}$</u> "square root trick".

We carry out the following moves:

(1) recenter.

The function $\varphi_\alpha \circ f : \mathcal{U} \longrightarrow \Delta$ omits the value $\varphi_\alpha(\alpha) = 0$ since f omits α. & $\varphi_\alpha \in \operatorname{Aut}\Delta$.

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(2) square root. Since $\mathcal{U}$ is a root domain & $\varphi_\alpha \circ f$ is nowhere zero, we can find $g : \mathcal{U} \longrightarrow \Delta$ holomorphic with $g^2(z) = \varphi_\alpha \circ f$.

Claim g injective.

Indeed $g(z) = g(w) \Rightarrow g(z)^2 = g(w)^2 \Rightarrow \varphi_\alpha \circ f(z) = \varphi_\alpha \circ f(w)$.
$\Rightarrow f(z) = f(w) \Rightarrow z = w$ since $f \in \mathcal{F}$ injective.

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(3) recenter. Let $\beta = g(a)$. We define

$$\tilde{f} = \varphi_\beta \circ g \Rightarrow \tilde{f}(a) = \varphi_\beta\, g(a) = \varphi_\beta(\beta) = 0.$$

& $\tilde{f} : \mathcal{U} \longrightarrow \Delta$ injective. Then $\tilde{f} \in \mathcal{F}$.

Outcome

$$g^2 = \varphi_\alpha \circ f, \quad \tilde{f} = \varphi_\beta \circ g, \quad \tilde{f} \in \mathcal{F}.$$

Comparison

$$g^2 = \varphi_\alpha \circ f \Rightarrow f = \varphi_{-\alpha} \circ g^2.$$

Let $s : \Delta \longrightarrow \Delta$, $s(w) = w^2 \Rightarrow f = \varphi_{-\alpha} \circ s \circ g.$

$$\Downarrow$$

$$\tilde{f} = \varphi_\beta \circ g \Rightarrow g = \varphi_{-\beta} \circ \tilde{f} \Rightarrow f = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \tilde{f}$$

Let $F : \Delta \longrightarrow \Delta$, $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}. \Rightarrow f = F \circ \tilde{f}$

Claim

$F \notin \operatorname{Aut} \Delta$, $F(0) = 0$.

Indeed, if $F \in \operatorname{Aut} \Delta$, $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \in \operatorname{Aut} \Delta$

$\Rightarrow s \in \operatorname{Aut} \Delta$. But s is not even injective as $s(z) = s(-z)$.

To see $F(0) = 0$ we compute

$$F(0) = \varphi_{-\alpha} \cdot s \cdot \varphi_{-\beta}(0) = \varphi_{-\alpha} \circ s(\beta) = \varphi_{-\alpha}(\beta^2) = \varphi_{-\alpha}(-\alpha) = 0$$

where we used

$$\beta^2 = g(a)^2 = \varphi_\alpha \circ f(a) = \varphi_\alpha(0) = -\alpha.$$

This is exactly what we needed to complete the proof of Step 3 & the proof of Riemann Mapping.

Remarks

$\boxed{1}$ Uniqueness of the biholomorphism. Take two biholom.

$$f, g : U \longrightarrow \Delta, \quad f(a) = g(a) = 0 \quad \text{then}$$

consider $\Delta \xrightarrow[f^{-1}]{} U \xrightarrow[g]{} \Delta$, $gf^{-1}(0) = 0$, $gf^{-1} \in \mathrm{Aut}\,\Delta$.

Then

$$gf^{-1} = \mathrm{Rot} \Rightarrow g = \mathrm{Rot} \circ f.$$

Thus the biholomorphisms we constructed are unique up to rotations.

$\boxed{ii}$ The extremal function f we constructed maximizes the derivatives at a of ALL functions $g : \mathcal{U} \to \Delta$, $g(a) = 0$ not only the INJECTIVE ones.

Indeed if $f : \mathcal{U} \to \Delta$ is the function we constructed, then $\forall g : \mathcal{U} \to \Delta$, $g(a) = 0$,

$$\Delta \xrightarrow[f^{-1}]{} \mathcal{U} \xrightarrow{g} \Delta \quad , \quad F = g \circ f^{-1} : \Delta \to \Delta .$$

$$F(0) = 0 .$$

Then $g = F \circ f \Rightarrow |g'(a)| = |F'(0)| \, |f'(a)| \leq |f'(a)|$

where we used $|F'(0)| \leq 1$ by Schwarz.

$\boxed{iii}$ $\mathcal{U}, V$ simply connected, $\mathcal{U}, V \neq \mathbb{C} \Rightarrow \mathcal{U}, V$ are biholomorphic. ($\mathcal{U} \cong \Delta \cong V$ transitive)

2. Loose ends

TFAE

$\boxed{i}$ $\mathcal{U}$ simply connected

$\boxed{ii}$ $\mathcal{U}$ is a "logarithm domain"

$\boxed{iii}$ $\mathcal{U}$ is a root domain

A "logarithm domain" is a domain where $\forall\, f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero, we can define

$\log f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

Proof

$\boxed{i} \Rightarrow \boxed{ii}$. Math 220A, PSet 4, Solution to #2.

$\boxed{ii} \Rightarrow \boxed{iii}$. Define $\sqrt{f} = \mathrm{Exp}\left(\frac{1}{2} \log f\right)$ for all $f : \mathcal{U} \to \mathbb{C}$ nowhere zero.

$\boxed{iii} \Rightarrow \boxed{i}$. If $\mathcal{U} = \mathbb{C} \Rightarrow \mathcal{U}$ simply connected

Let $\mathcal{U} \neq \mathbb{C}$ $\Rightarrow$ let $f : \mathcal{U} \longrightarrow \Delta$, $g : \Delta \longrightarrow \mathcal{U}$ inverse biholomorphisms. If γ is a loop in $\mathcal{U}$, then

$f \circ \gamma$ loop in Δ = simply connected $\Rightarrow$ $f \circ \gamma \overset{\Delta}{\sim} 0$

$$\Rightarrow g \circ f \circ \gamma \overset{\mathcal{U}}{\sim} g(0) \Rightarrow \gamma \overset{\mathcal{U}}{\sim} g(0) \Rightarrow \gamma \text{ null homotopic.}$$

Question How do we construct biholomorphism $f : \mathcal{U} \longrightarrow \Delta$ explicitly?

Answer : depends on $\mathcal{U}$.

Some examples worth knowing

$\boxed{a}$ Lecture 11 :

$$\mathbb{C}^- \longrightarrow \Delta, \quad z \longrightarrow \left(\frac{1+z}{1-z}\right)^2.$$

We will give more examples next time.

In HWK 5, we will see a few more:

$\boxed{ii}$ $\underbrace{S = \{-1 < \operatorname{Re} z < 1\}}_{\text{strip}} \xrightarrow{\sim} \Delta$

$\boxed{iii}$ $\underbrace{\Delta \setminus (-1, 0]}_{\text{slit unit disc}} \xrightarrow{\sim} \Delta$

$\boxed{IV}$ $\underbrace{\Delta^+}_{\text{upper half disc}} \xrightarrow{\sim} \Delta$

$\boxed{V}$ $\mathbb{C} \setminus \{x \in \mathbb{R} : |x| \geq 1\} \xrightarrow{f^{-1}} \mathfrak{h}^+ \xrightarrow{c} \Delta$

where $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ $\rightsquigarrow$ 220A, HWK 2.