Math 220 B – Lecture 13

February 28, 2024

Last time

Riemann Mapping Theorem

$\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow$ $\mathcal{U}$ is biholomorphic to Δ.

Extension to the boundary

Question Given $f : \mathcal{U} \longrightarrow \Delta$ biholomorphism, does it extend $\bar{f} : \bar{\mathcal{U}} \longrightarrow \bar{\Delta}$ bicontinuously?

Answer $\boxed{i}$ yes if $\mathcal{U}$ bounded & $\partial \mathcal{U}$ = simple closed curve.

Caratheodory's theorem

$\boxed{ii}$ We will not give the proof in this course.

Beyond the boundary

Question Can we extend beyond the boundary?

The easiest instance is provided by

Schwarz Reflection Principle Conway IX. 1.

There are several versions but two stand out:

[i] reflection across line segments (book)

[ii] reflection across circular arcs (HWK5).

Applications

[i] biholomorphic maps between rectangles, annuli

[ii] analytic continuation ...

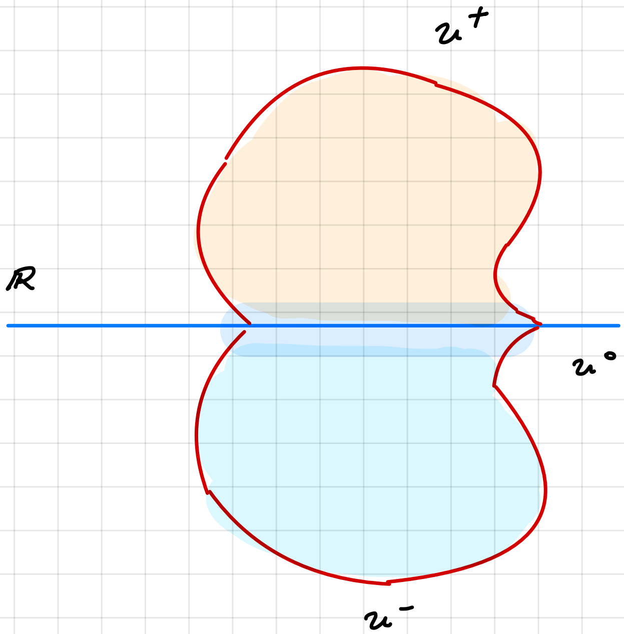
Reflection across segments

open $\mathcal{U} \subseteq \mathbb{C}$ symmetric $z \longrightarrow \bar{z}$. $\forall z \in \mathcal{U} \Rightarrow \bar{z} \in \mathcal{U}$.

$\mathcal{U}^+ = \mathcal{U} \cap \mathfrak{h}^+$

$\mathcal{U}^- = \mathcal{U} \cap \mathfrak{h}^-$

$\mathcal{U}^0 = \mathcal{U} \cap \mathbb{R}$.



Given $f : \mathcal{U}^+ \longrightarrow \mathbb{C}$

[i] holomorphic in $\mathcal{U}^+$

[ii] extends continuously to $\mathcal{U}^0$.

[iii] such that the values $f(\mathcal{U}^0) \subseteq \mathbb{R}$.

Define

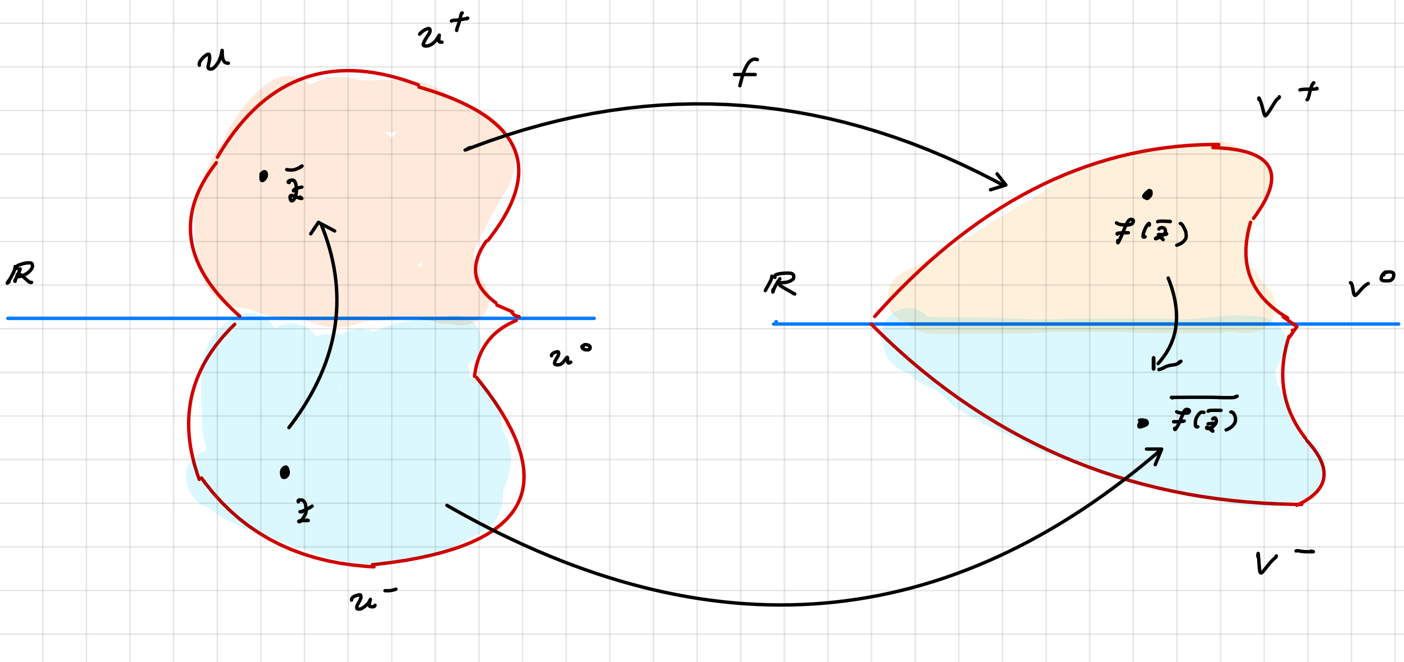
$$F(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U}^+ \\ f(z) & \text{if } z \in \mathcal{U}^0 \\ \overline{f(\bar{z})} & \text{if } z \in \mathcal{U}^- \end{cases}$$

Theorem The function $F : \mathcal{U} \longrightarrow \mathbb{C}$

is a holomorphic extension of f (beyond $\mathcal{U}^0$ which is part of the boundary).

Remarks

[1] Visualization

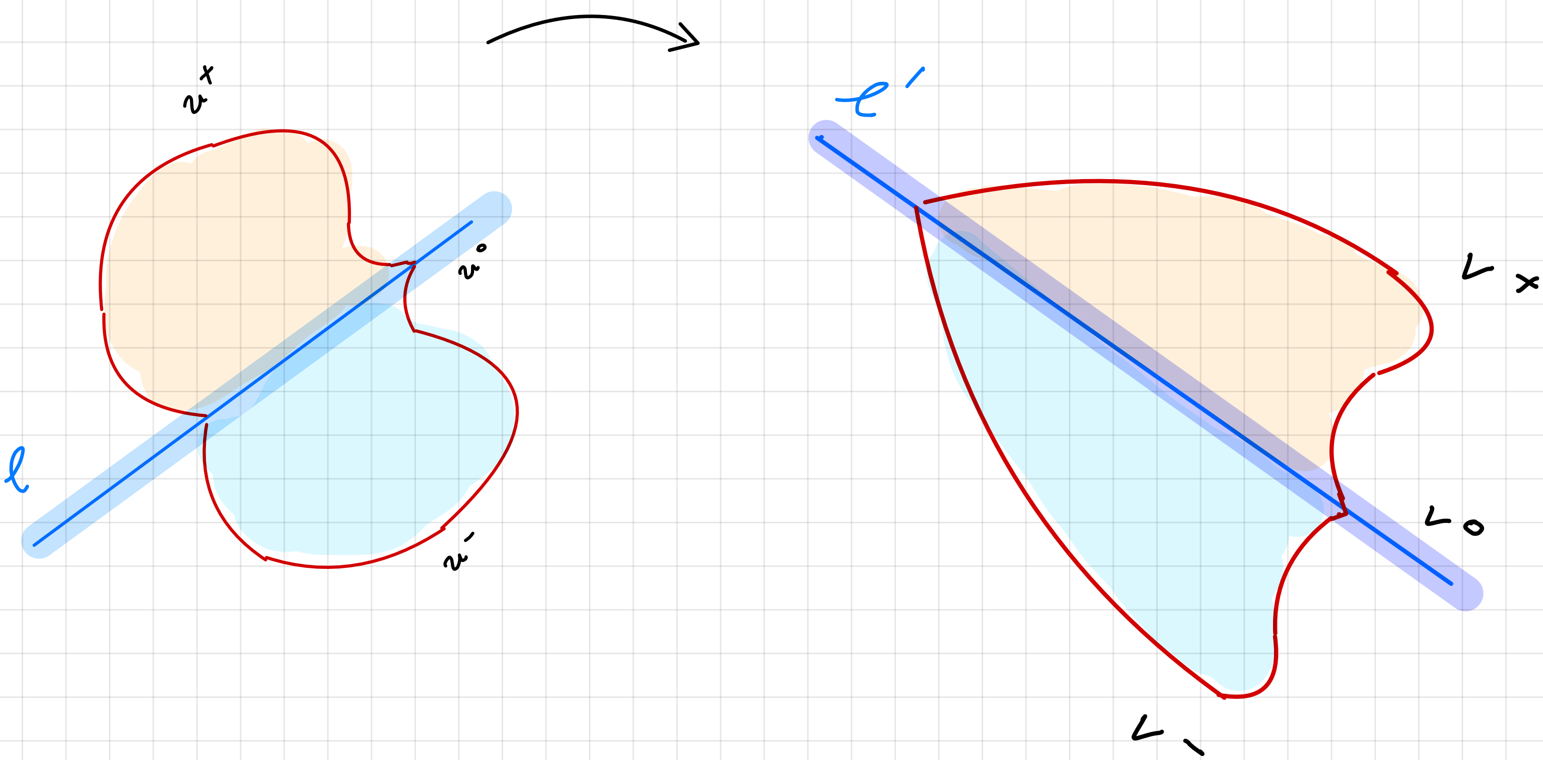


$\boxed{11}$ The condition

$$f(U_0) \subseteq \mathbb{R}$$

ensures we reflect across real axis on both sides.

More generally, we can reflect across arbitrary lines



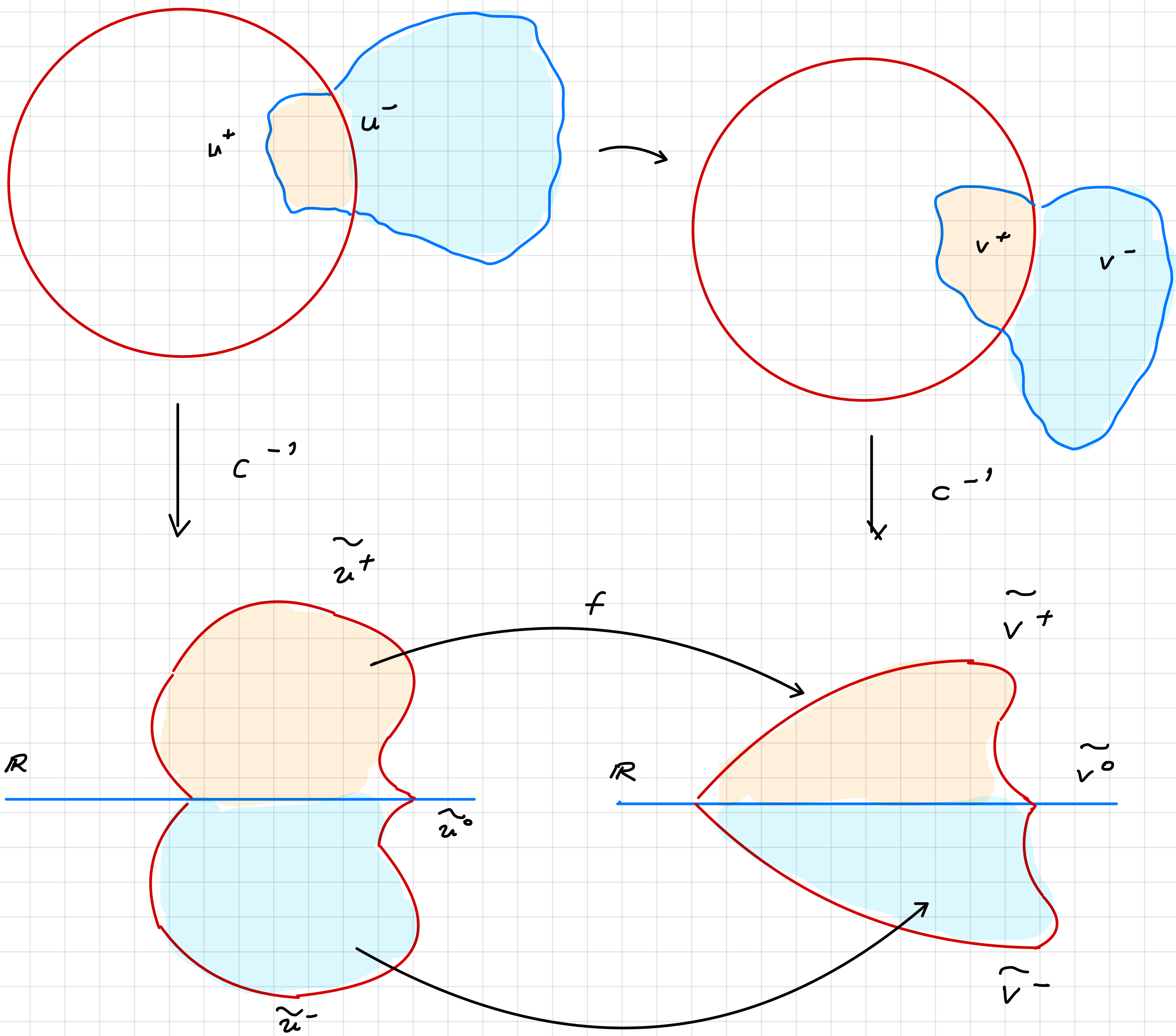
This can be deduced via rotations

$\boxed{iii}$ Using the Cayley transform

$$C : \Delta \longrightarrow \mathfrak{h}^{+}$$

we can also reflect across arcs in the unit disc.

(HWK 5).



Proof of Schwarz

$\boxed{I}$ F continuous

$\boxed{II}$ F holomorphic in U^+

$\boxed{III}$ F holomorphic in U^-

$\boxed{IV}$ F holomorphic at points of U^0.

Proof of $\boxed{I}$

Let $z_0 \in U_0 \Rightarrow z_0 = \overline{z_0}$

We show $\displaystyle\lim_{\substack{z \to z_0 \\ z \in U^+}} F(z) = \lim_{\substack{z \to z_0 \\ z \in U^-}} F(z)$.

$$\Longleftrightarrow \quad \lim_{\substack{z \to z_0 \\ z \in U^+}} f(z) = \lim_{\substack{z \to z_0 \\ z \in U^-}} \overline{f(\overline{z})}$$

$$\Longleftrightarrow \quad f(z_0) = \overline{f(\overline{z_0})}$$

which holds since $z_0 = \overline{z_0}$ & $f(z_0) = \overline{f(z_0)}$

Proof of $\boxed{iii}$ We show F holomorphic in $\mathcal{U}^-$.

Let $c^- \in \mathcal{U}^-$. Let $c^+ = \overline{c^-} \in \mathcal{U}^+$. Since f is holomorphic at $c^+ \Rightarrow \exists\ \Delta(c^+, r) \subseteq \mathcal{U}^+$. Taylor expand in $\Delta(c^+, r)$:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - c^+)^k, \quad \text{radius of convergence} \geq r.$$

Let $z \in \Delta(c^-, r) = \overline{\Delta(c^+, r)}$. Then

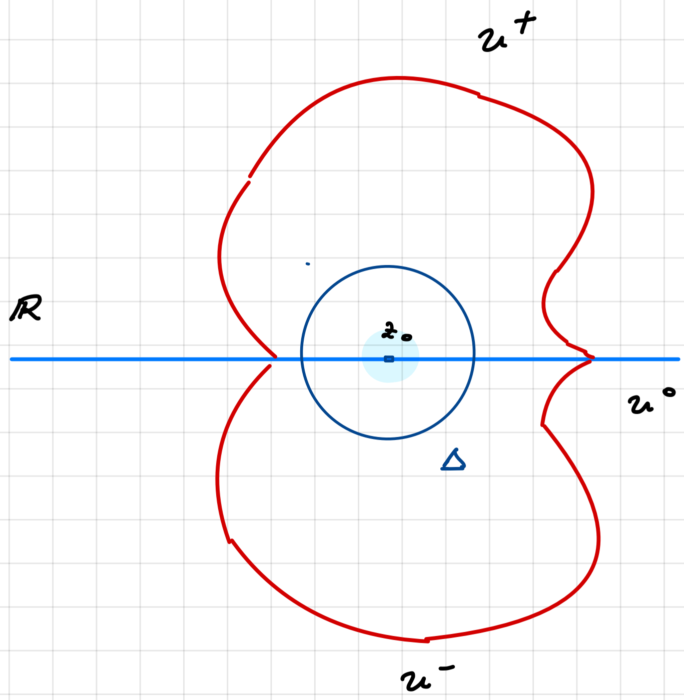
$$F(z) = \overline{f(\bar{z})} = \overline{\sum_{k=0}^{\infty} a_k (\bar{z} - c^+)^k}$$

$$= \sum_{k=0}^{\infty} \overline{a_k} (z - \overline{c^+})^k$$

$$= \sum_{k=0}^{\infty} \overline{a_k} (z - c^-)^k, \quad \text{radius of convergence} \geq r.$$

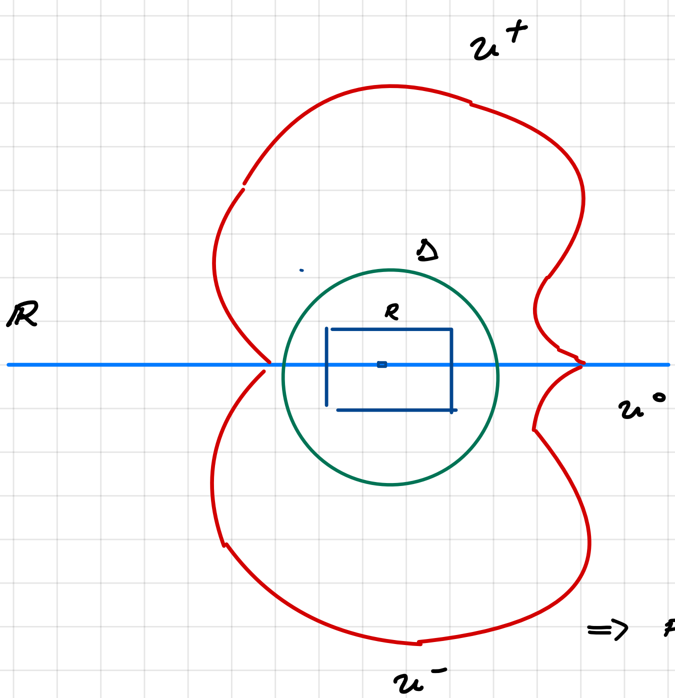
$\Rightarrow F$ holomorphic in $\mathcal{U}^-$.

Proof of $\boxed{IV}$



We show F is holomorphic in discs $z_0 \in \Delta \subseteq \mathcal{U}$ for arbitrary $z_0 \in \mathcal{U}_0$.

This will complete the proof.



Goal: $\forall\ \overline{R} \subseteq \Delta$

$$\int_{\partial R} F\, dz = 0$$

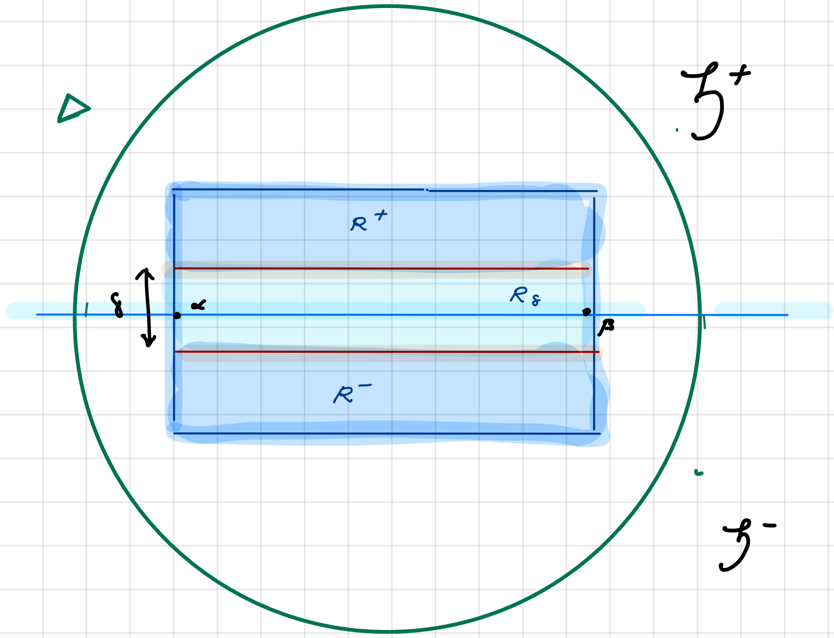
By Math 220A, Lecture 4

$\Rightarrow F = G'$ for some holomorphic G in Δ

$\Rightarrow F$ holomorphic in Δ.

If $\bar{R} \subseteq \mathcal{U}^+$ or $\bar{R} \subseteq \mathcal{U}^-$ this is clear (Goursat / Cauchy).

Assume $\bar{R}$ intersects the real axis. We assume that



the intersection is not a side of R. Otherwise the argument is simpler.

We show $\exists\ K > 0$ such that for all $\varepsilon > 0$,

$$\left| \int_{\partial R} F\, dz \right| \leq K \cdot \varepsilon. \implies \int_{\partial R} F\, dz = 0.$$

$\boxed{I}$ F continuous in $\bar{\Delta}$ $\implies$ $|F(z)| \leq M$ for all $z \in \bar{\Delta}$.

$\boxed{II}$ F uniformly continuous in $\bar{\Delta}$ = compact.

$\implies \forall\ \varepsilon\ \exists\ \delta$, $|x-y| \leq \delta \implies |F(x) - F(y)| < \varepsilon$.

We may assume $\delta < \varepsilon$.

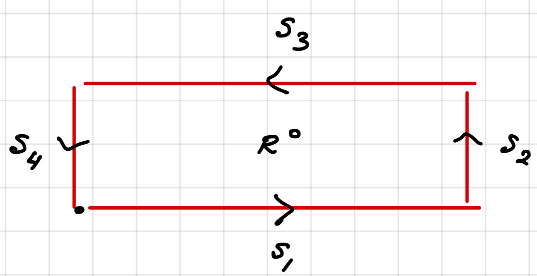
[i] Construct R^+, R^-, R^0 when $R^+ \subseteq \mathcal{U}^+$, $R^- \subseteq \mathcal{U}^-$

$$R^0 = [\alpha, \beta] \times \left[-\frac{\delta}{2}, \frac{\delta}{2}\right].$$

[ii] $\displaystyle\int_{\partial R^+} F\, dz = 0$, $\displaystyle\int_{\partial R^-} F\, dz = 0$ by Goursat.

$$\Rightarrow \int_{\partial R} F\, dz = \int_{\partial R^0} F\, dz.$$

Estimates:



Sides of R^0: S_1, S_2, S_3, S_4.

(1) $$\left| \int_{S_2} F\,dz + \int_{S_4} F\,dz \right| \leq \left| \int_{S_2} F\,dz \right| + \left| \int_{S_4} F\,dz \right|$$

$$\leq M \cdot \underbrace{\text{length } S_2}_{\delta} + M \cdot \underbrace{\text{length } S_4}_{\delta}$$

$$= 2M\delta < 2M\varepsilon.$$

parametrize

(2) $$\left| \int_{S_1} F\,dz + \int_{S_3} F\,dz \right| \leq \int_{\alpha}^{\beta} \underbrace{\left| F\left(t - \frac{i\delta}{2}\right) - F\left(t + \frac{i\delta}{2}\right) \right|}_{<\varepsilon} dt$$

(uniform continuity).

$$\leq \varepsilon \cdot (\beta - \alpha) \leq \varepsilon \cdot \text{diam}(\Delta)$$

(1)+(2)

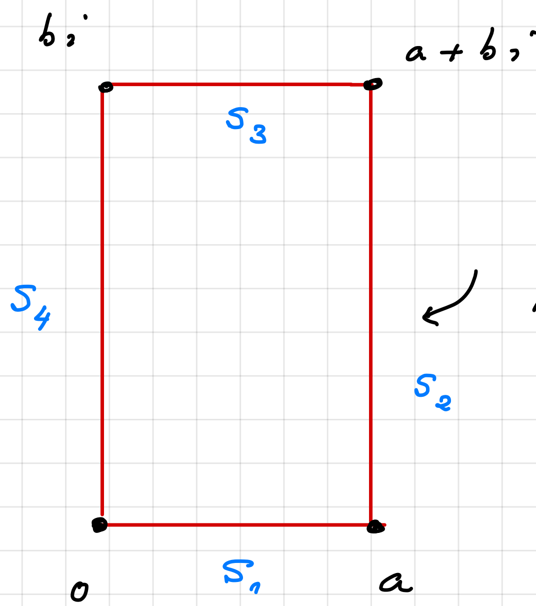
$$\Rightarrow \left| \int_{\partial R^\circ} F\,dz \right| \leq \left| \int_{S_2} F\,dz + \int_{S_4} F\,dz \right| + \left| \int_{S_1} F\,dz + \int_{S_3} F\,dz \right|$$

$$\leq 2M\varepsilon + \varepsilon \cdot \text{diam}(\Delta) = K\varepsilon.$$

This completes the proof.

2. Application

Conformal maps of rectangles



$R_{a,b}$, $a,b \in \mathbb{R}$.

Example

$\exists$ biholomorphism $f : R_{a,b} \longrightarrow R_{a',b'}$ such that

[i] f extends continuously & bijectively to the boundary.

[ii] sending corners to corners & edges to edges.

IF AND ONLY IF

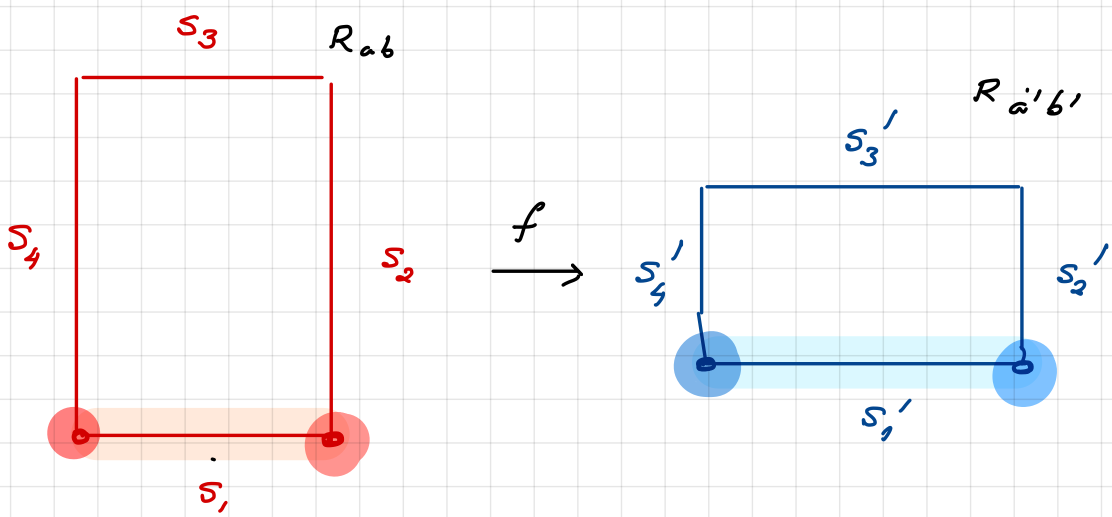
$$\frac{a'}{a} = \pm \frac{b'}{b} \quad \text{or} \quad a a' = \pm b b'$$

Remark Condition $\boxed{ii}$ is automatic by Caratheodory,

while condition $\boxed{iii}$ is really necessary.

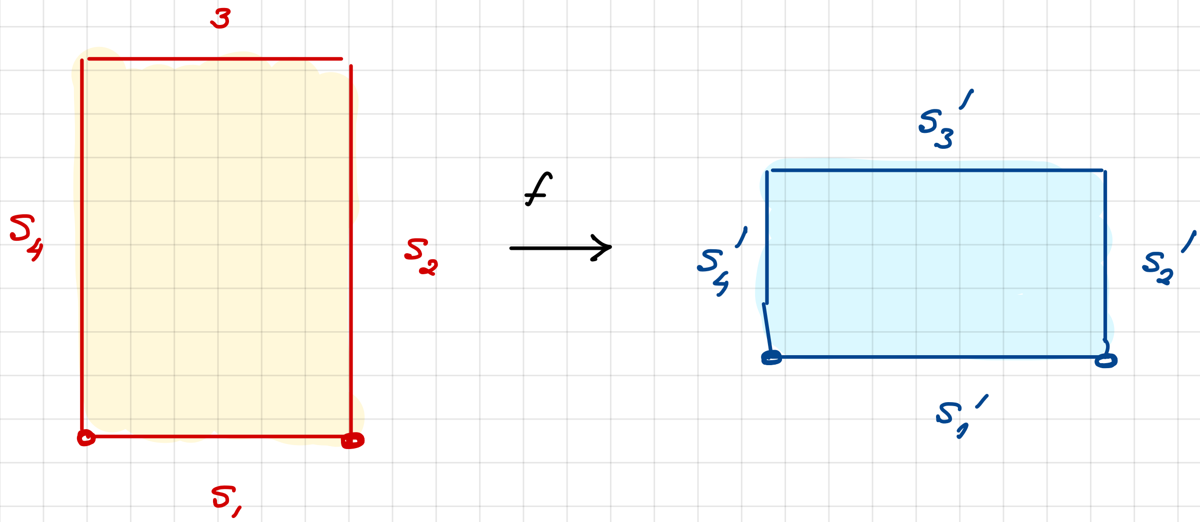
We first assume

$$f: S_1 \longrightarrow S_1', \quad 0 \longrightarrow 0, \quad a \longrightarrow a'.$$

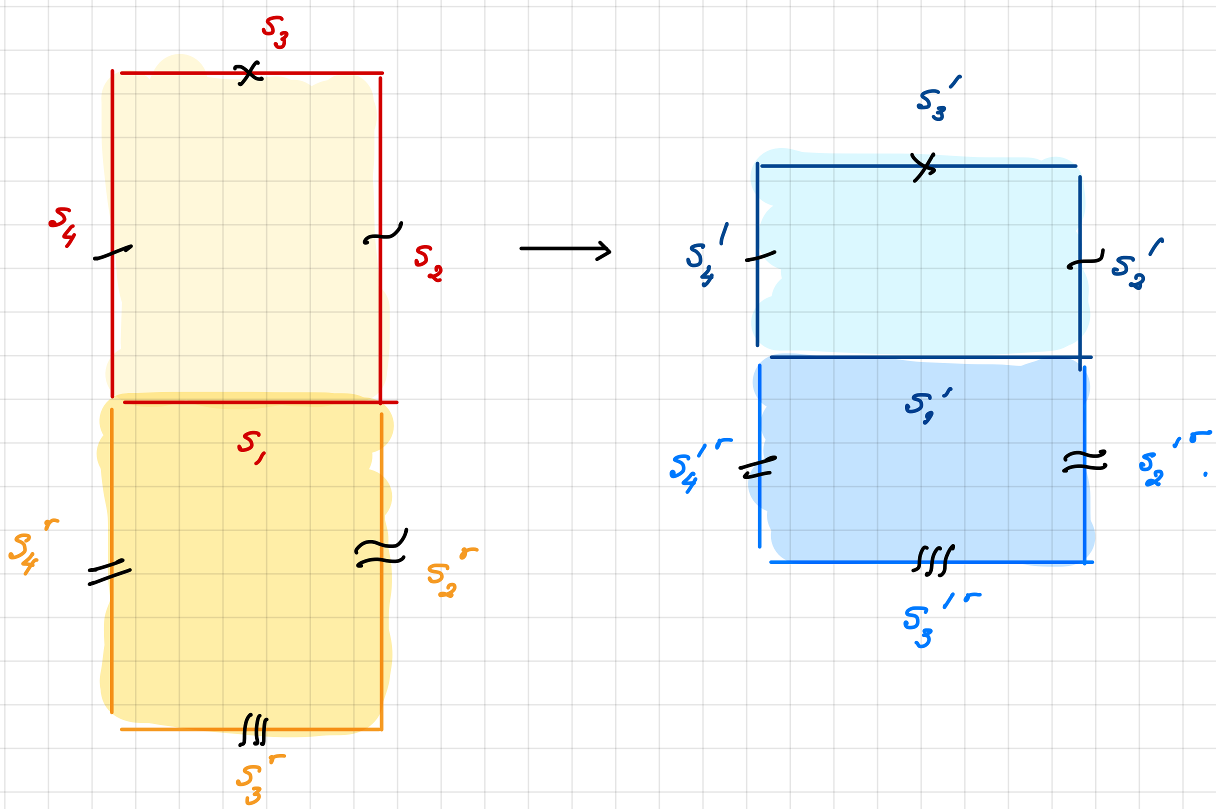


$f(0) = 0, \quad f(a) = a'$

- S_4 is sent to a side containing $f(0)=0$, hence S_4'
- S_2 is sent to a side containing $f(a)=a'$, hence S_2'
- S_3 is sent to the remaining side S_3'



We use Schwarz Reflection along S_1 & S_1'!



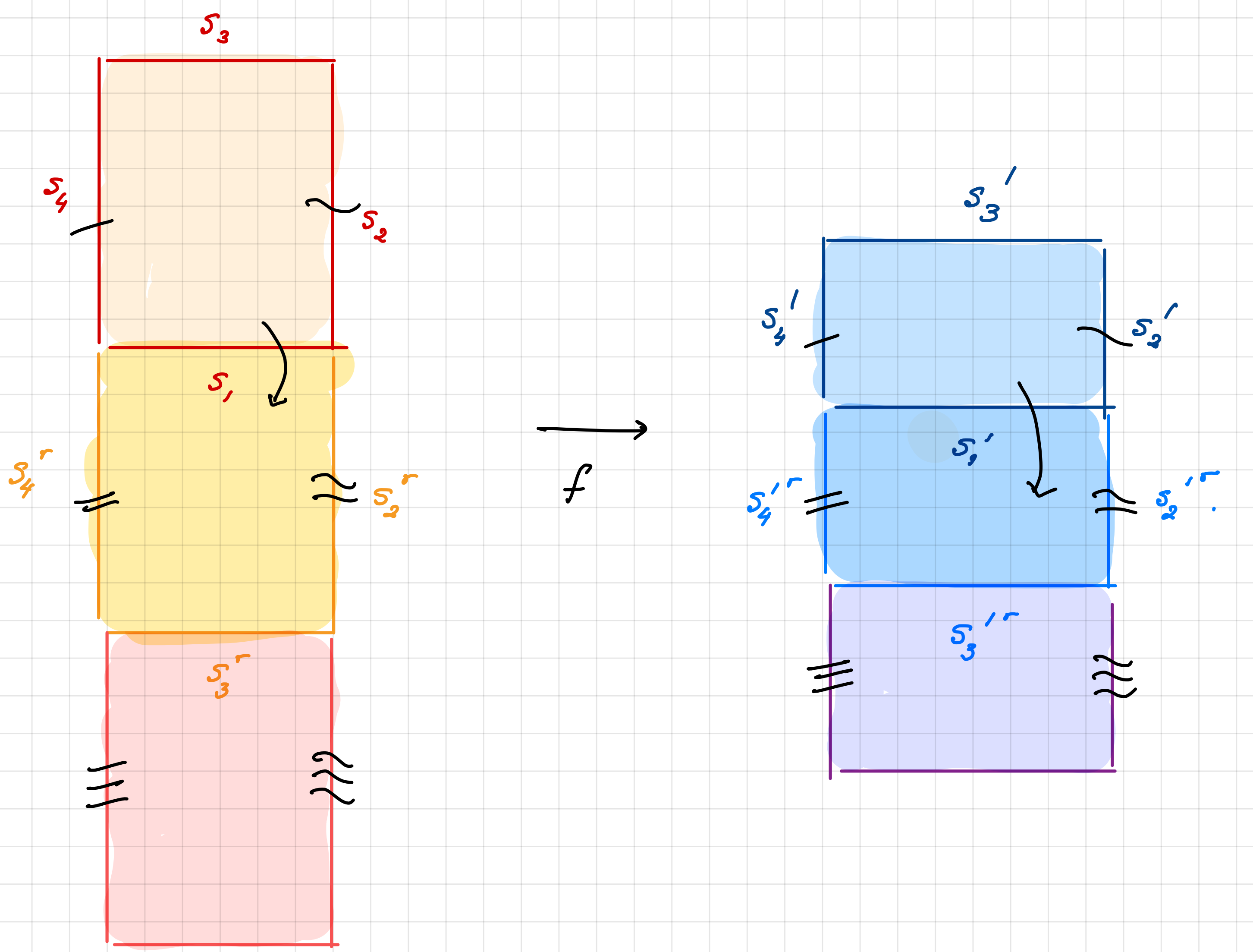
Note

$$S_4^{r} \longrightarrow S_4'^{r}, \quad S_2^{r} \longrightarrow S_2'^{r}, \quad S_3^{r} \longrightarrow S_3'^{r}.$$

from the explicit formula for the extension

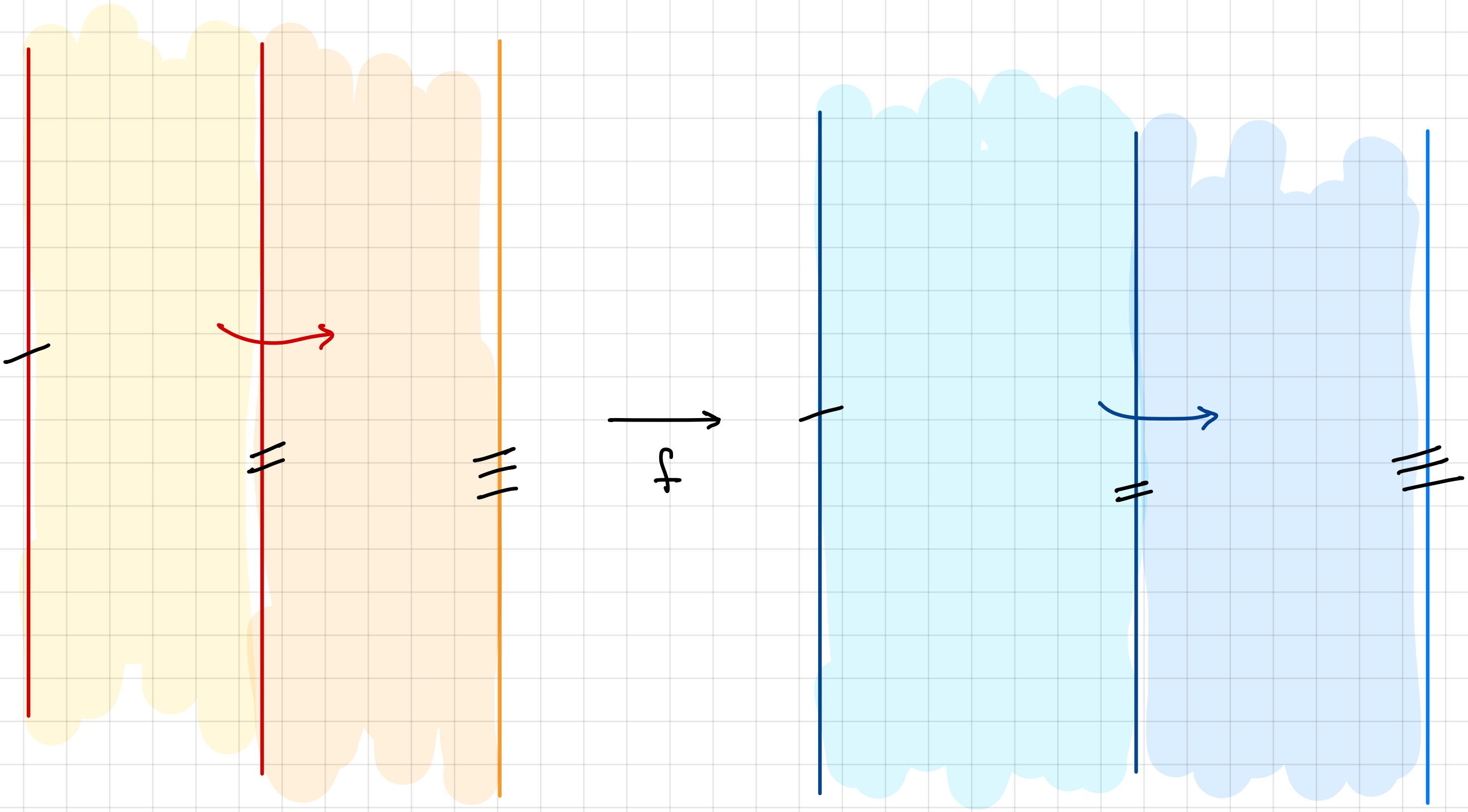
The extension is still bijective. (as the picture shows).

Reflect the new rectangle one more time, across S_3^r & $S_3'^r$.



and continue until we get two strips mapping to each other & their boundaries are mapped to each other.

Now reflect the strips across their sides.



In the end, we obtain $f: \mathbb{C} \longrightarrow \mathbb{C}$ bijective & holomorphic.

We saw in Math 220A, PSet 5 that $f(z) = \alpha z + \beta$.

Since $f(0) = 0 \implies \beta = 0 \implies f(z) = \alpha z$.

$$\begin{aligned} f(a) &= a' \implies \alpha a = a' \\ f(b) &= b' \implies \alpha b = b' \end{aligned} \implies \frac{a'}{a} = \frac{b'}{b}.$$

The remaining cases are similar.