Math 220 B — Lecture 14

March 4, 2024

Part I : Weierstraß & Mittag-Leffler

Series & Products

Part II Riemann & Schwarz

Mapping Theory

Part III Runge ← Conway VIII. 1.

Approximation theory

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§1. Context for Runge

In real analysis (Math 140B), we learn

Weierstraß Approximation theorem

$f : [a,b] \longrightarrow \mathbb{R}$ continuous, $\exists$ P_n polynomials

$$P_n \rightrightarrows f.$$

This was proven by Weierstraß at age 70 in 1885.

There are many applications of this theorem.

e.g. in Fourier analysis, functional analysis etc.

Remark This can be generalized in $\mathbb{R}^n$.

If $K \subseteq \mathbb{R}^n$ compact, $f : K \longrightarrow \mathbb{R}$ continuous, then

$$\exists\ P_n \text{ polynomials, } P_n \rightrightarrows f \text{ in } K.$$

Runge (age 29, Ph. D. 1880, student of Weierstraß):

Question What about f holomorphic? Can it be approximated by polynomials in z?

Answer was given in 1885 as well.

Remark This doesn't follow from Weierstraß.

Weierstraß produces polynomials in x, y for $z = x+iy$.

e.g. polynomials in z and $\bar{z}$.

ZUR THEORIE DER EINDEUTIGEN ANALYTISCHEN FUNCTIONEN

VON

C. RUNGE[1]
in BERLIN.

Seit dem Bekanntwerden der Modulfunctionen, weiss man, dass der Gültigkeitsbereich einer analytischen Function nicht nothwendig von discreten Punkten begrenzt zu sein braucht, sondern dass auch continuirliche Linien als Begrenzungsstücke auftreten und einen Theil der complexen Ebene von dem Gültigkeitsbereich ausschliessen können.

Hier entsteht nun die Frage, ob der Gültigkeitsbereich analytischer Functionen seiner Form nach irgend welchen Beschränkungen unterliegt oder nicht. Diese Frage bildet, so weit sie sich auf eindeutige analytische Functionen bezieht, den Gegenstand der nachfolgenden Untersuchung. Es wird sich ergeben, dass der Gültigkeitsbereich einer eindeutigen analytischen Function d. h. die Gesammtheit aller Stellen an denen sie sich regulär oder ausserwesentlich singulär verhält keiner andern Beschränkung unterliegt als derjenigen, zusammenhängend zu sein. In dem ersten Theile

[1] Die Aufgabe, welche in dem ersten Paragraphen dieser Arbeit in eleganter Weise gelöst wird, ist nicht in meiner Abhandlung *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante* (Acta mathematica 4, S. 1—79) behandelt worden. Diejenige Aufgabe dagegen, mit welcher sich der Verfasser in dem zweiten Paragraphen beschäftigt, ist in meiner Abhandlung aus mehreren verschiedenen Gesichtspunkten betrachtet und gelöst worden. Da jedoch der Verfasser seine Untersuchungen vor der Veröffentlichung meiner oben citirten Abhandlung machte und auch ganz andere mit dem CAUCHY'schen Integralsatze in Zusammenhang stehende Methoden braucht, so habe ich die ganze Arbeit für geeignet gehalten hier aufgenommen zu werden.

Der Herausgeber.

Acta mathematica. 6. Imprimé 29 Septembre 1884.

Acta Math 6 (1885)

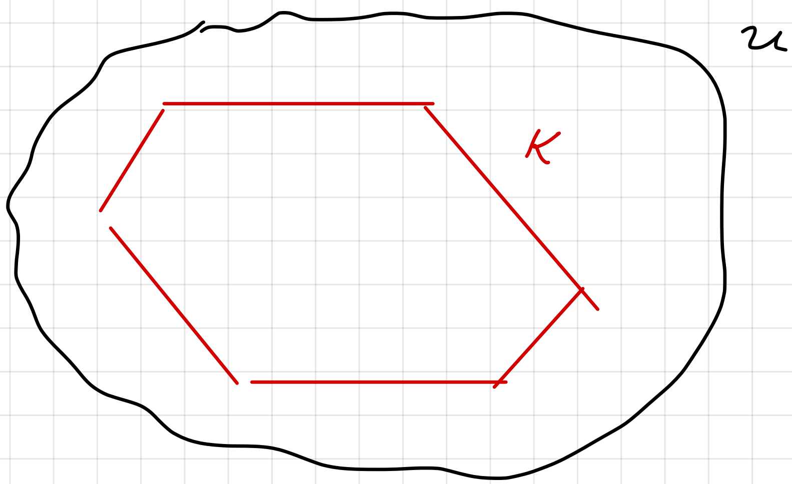
Carl Runge (1856 - 1927)

- Runge - Kutta
- Runge's Approximation
- mathematics, astrophysics, spectroscopy.

§2. Pharsing the Question more carefully

Beware: A holomorphic function is defined over OPEN sets. (see Math 220A).

Definition $K \subseteq \mathbb{C}$ compact. A holomorphic function in K is a function $f : K \longrightarrow \mathbb{C}$ that extends holomorphically to a neighborhood $U \supseteq K$.



Two versions of the question

Runge C (compact sets) $\quad K \subseteq \mathbb{C}$ compact

Given f holomorphic in K, are there polynomials P_n such that $P_n \rightrightarrows f$ in K ?

Runge O (open sets) $\quad \mathcal{U} \subseteq \mathbb{C}$ open

Given f holomorphic in $\mathcal{U}$, are there polynomials P_n such that $P_n \overset{\text{l.u.}}{\rightrightarrows} f$ in $\mathcal{U}$?

Emphasis

Runge $\mathbb{C}$: approximation on a single compact K

Runge $\mathcal{O}$: approximation on all compacts K

in the domain of a holomorphic function

Runge $\mathbb{C}$ is more basic.

complex analysis

Runge $\mathbb{C}$ $\Longrightarrow$ Runge $\mathcal{O}$

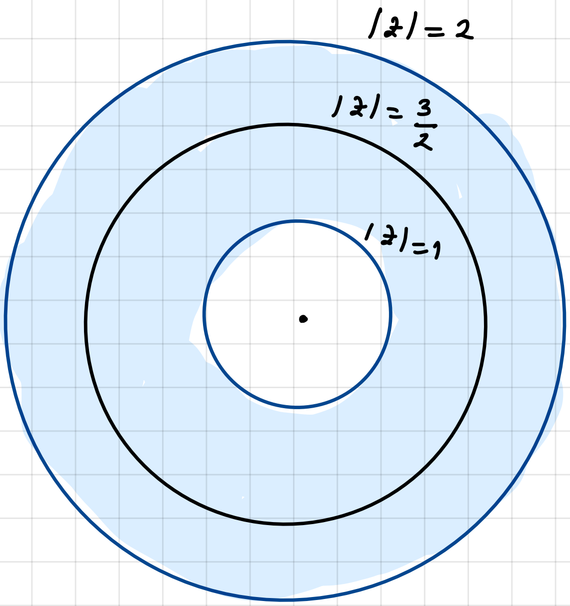
point-set topology.

The two versions are very similar.

Example   Runge C.

$K = \{ 1 \le |z| \le 2 \}$, $\quad f(z) = \frac{1}{z}$. holomorphic in K.

Can we find $P_n \rightrightarrows f$ in K ?



No ! Note f is holomorphic in

$\mathcal{U} \supseteq K$, $\mathcal{U} = \{ \frac{1}{2} < |z| < \frac{5}{2} \}$

so "holomorphic in K".

If $P_n \rightrightarrows f$ in K then

$$\int_{|z| = \frac{3}{2}} P_n \, dz \longrightarrow \int_{|z| = \frac{3}{2}} f \, dz.$$

Note $\displaystyle\int_{|z| = \frac{3}{2}} P_n \, dz = 0$ & $\displaystyle\int_{|z| = \frac{3}{2}} f \, dz = 2\pi i$ by the

residue theorem. This is a contradiction.

The failure is due to the "hole" in K. & the fact f has a pole there

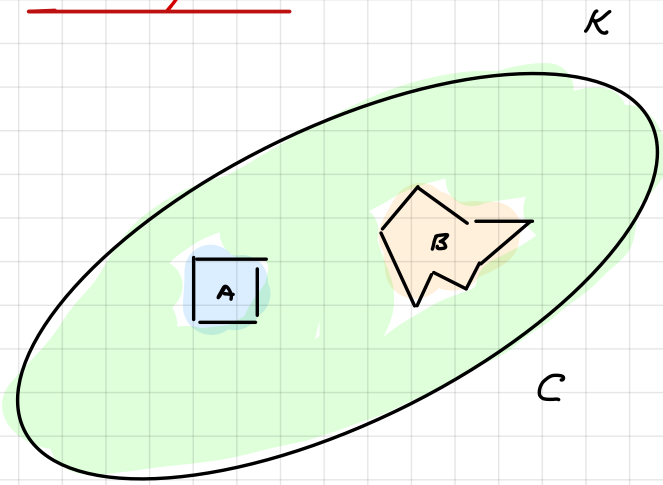
What is a "hole"?

Definition $K \subseteq \mathbb{C}$ compact

A hole is a bounded connected component of $\mathbb{C} \setminus K$.

Example

$\boxed{I}$



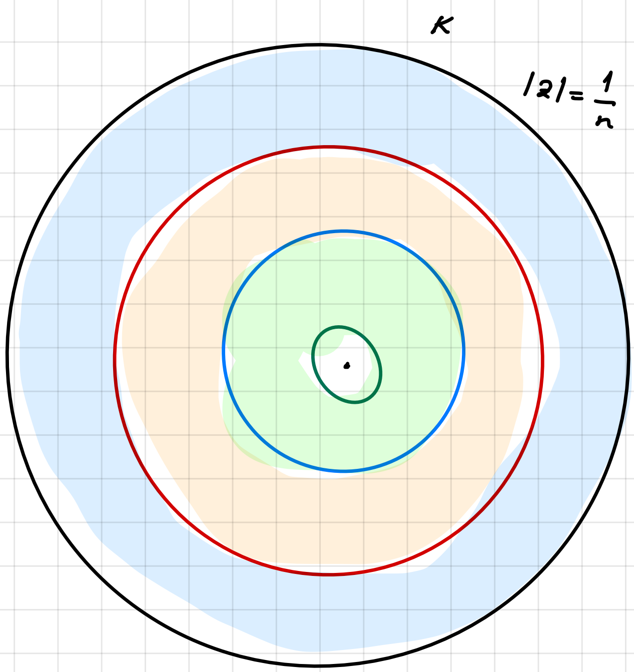
$\mathbb{C} - K = A \cup B \cup C$

C unbounded

A, B bounded

A, B are holes for K.

$\boxed{II}$



$K = \bigcup_{n \geq 1} \{ |z| = \frac{1}{n} \} \cup \{0\}$

K closed & bounded $\Rightarrow$

$\Rightarrow K$ compact.

∞-many holes

$H_n = \{ \frac{1}{n+1} < |z| < \frac{1}{n} \}$.

§ 3. Runge's Theorem - Compact Sets

We give three versions. The simplest version is:

Runge's little theorem (Case $\mathbb{C}$)

If K has no holes ($\iff \mathbb{C} \setminus K$ connected)

then $\forall f$ holomorphic in K, $\exists$ polynomials P_n

with

$$P_n \rightrightarrows f \text{ in } K.$$

Question How about arbitrary K ?

Answer Polynomial approximation fails (Example)

Are we even asking the right question?

Better Rational Approximation.

Question Given f holomorphic in K,

$\exists$ R_n rational functions, $R_n \rightrightarrows f$ in K &

poles of R_n are outside K ?

Question Can we prescribe the location of the poles of R_n ?

Runge C (Almost final) $K \subseteq \mathbb{C}$ compact.

Thm Let S be a set of points,

at least one from each hole of K.

then $\forall f$ holomorphic in K.

i) $\exists\ R_n \rightrightarrows f$ in K

ii) R_n are rational functions whose poles are $\overbrace{\text{in } S}^{\text{in } \mathbb{C}}$.

Remark The poles of R_n are contained in S, but it may happen that not all points of S are poles.

Remark If K has no holes then $S = \emptyset$. Thus

R_n has no poles $\Rightarrow$ R_n have no denominators $\Rightarrow$

$\Rightarrow$ R_n are polynomials. We recover Little Runge.

Runge C Final Form

Conway VIII. 1. 7.

We replace $\mathbb{C}$ by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Thm Let $K \subseteq \mathbb{C}$, compact. Let $S \subseteq \hat{\mathbb{C}}$ be a set of points,

at least one chosen from each component of $\hat{\mathbb{C}} \setminus K$.

Let f be holomorphic in K. Then

[i] $\exists\ R_n \rightrightarrows f$ in K

in $\hat{\mathbb{C}}$

[ii] R_n are rational with possible poles in S.

Remark An interesting case allowed by the Final Version is to pick $\infty \in S$ from the unbounded component.

Thus, when S consists in

- ∞ from the unbounded component of $\hat{\mathbb{C}} \setminus K$
- a point from each bounded component of $\mathbb{C} \setminus K$ (holes)

we recover Almost Final Version.

The two versions are even equivalent in this case since the condition that a rational function R have at worst a pole at ∞ is vacuous. Indeed,

$$R(z) = \frac{\prod_{i=1}^{n} (z - a_i)}{\prod_{i=1}^{m} (z - b_i)} \Rightarrow R\left(\frac{1}{z}\right) = z^{m-n} \frac{\prod (1 - a_i z)}{\prod (1 - b_i z)}$$

has at worst a pole at 0.

Summary

Runge C (Final) $\Longrightarrow$ Runge C (Almost Final)

Conway VIII.1.7

- rational approximation
- version for $\hat{\mathbb{C}}$

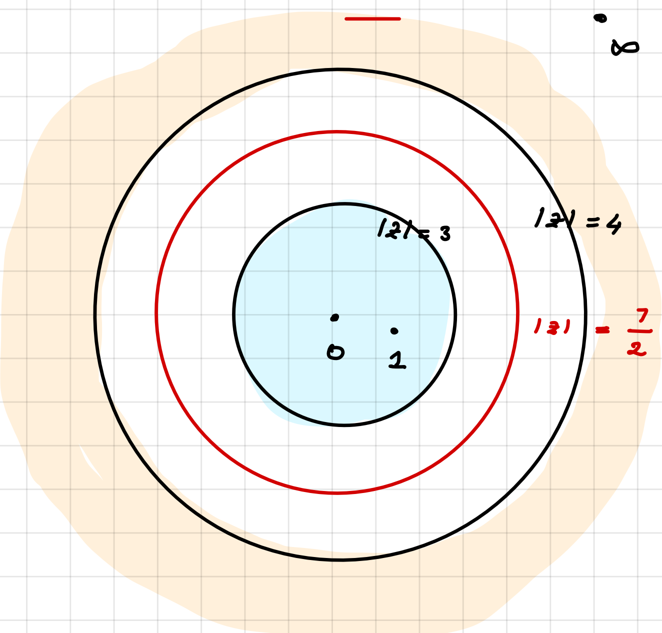
- rational approximation
- poles in each hole

Little Runge C

- polynomial approximation
- K has no holes

Example / Review

$$f(z) = \frac{z^3}{(z-2)(z-7)}$$



$K = \{3 \le |z| \le 4\}$.

f is holomorphic in K because it extends holomorphically to

$\mathcal{U} = \{\frac{5}{2} < |z| < \frac{9}{2}\} \supseteq K$.

Can we approximate f uniformly on K by:

(1) rational functions with poles in $\mathbb{C}$ at 1 ?

YES Amost Final Version. Poles in $\hat{\mathbb{C}}$ are $1, \infty$.

(2) rational functions with poles at $0, \infty$

YES Final Version

(3) rational functions with poles at ∞ ?

NO. Such rational functions would have to be polynomials (if they had denominators, there would be poles). But if $p_n \rightrightarrows f$ then

$$\int_{|z|=\frac{7}{2}} p_n \, dz \longrightarrow \int_{|z|=\frac{7}{2}} f \, dz = 2\pi i \; \mathrm{Res}(f, 2)$$

$$\int_{|z|=\frac{7}{2}} p_n \, dz = 0$$

$$= 2\pi i \cdot \left. \frac{z^3}{z-7} \right|_{z=2} \neq 0$$

using the Residue theorem. Contradiction!

How about the converse?

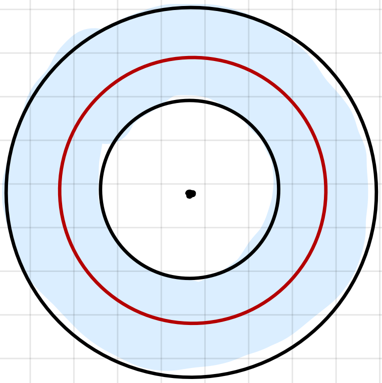
If K has no holes $\overset{\text{Runge}}{\Rightarrow}$ polynomial approximation holds.

If K has holes $\Rightarrow$ polynomial approximation fails in general

How to see this?

Two methods

$\boxed{\text{i}}$ $K = \{1 \leq |z| \leq 2\}$, $f(z) = \dfrac{1}{z}$.



If $P_n \rightrightarrows f$ in K, P_n polynomials

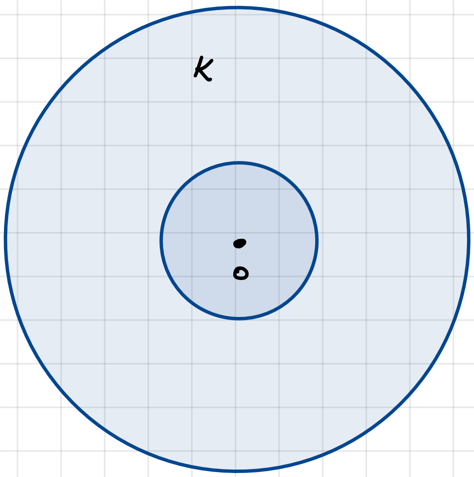
then $\underbrace{\int_{|z|=\frac{3}{2}} P_n \, dz}_{0} \longrightarrow \int_{|z|=\frac{3}{2}} f \, dz = \underbrace{\int_{|z|=\frac{3}{2}} \frac{dz}{z}}_{2\pi i}$

Both integrals follow by the residue theorem, for instance.

This contradiction shows f cannot be approximated uniformly in K by polynomials P_n.

$\boxed{ii}$ (New method).

$$K = \{1 \le |z| \le 2\}, \quad f(z) = \frac{1}{z}$$



Assume $P_n \rightrightarrows f$ in K, P_n polynomials.

$$\exists N : \quad |P_N - f| < \frac{1}{4} \quad \text{on } K$$

$$\iff \left|P_N - \frac{1}{z}\right| < \frac{1}{4} \quad \text{on } K.$$

$$\iff |z P_N - 1| < \frac{|z|}{4} \quad \text{on } K.$$

$\Rightarrow |z P_N - 1| < \frac{|z|}{4}$ when $|z| = 1$. $\Rightarrow |z P_N - 1| < \frac{1}{4}$. when $|z| = 1$.

Let $g(z) = 1 - z P_N \Rightarrow g$ entire. Note $|g(0)| = 1$ and

$$|g(z)| < \frac{1}{4} \quad \text{for } |z| = 1.$$

This contradicts Maximum modulus for g in $\overline{\Delta}(0,1)$.

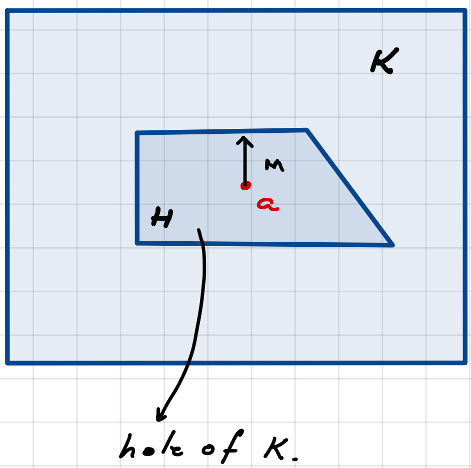
The second method generalizes

Let H be a hole of K. Let $a \in H$., $f(z) = \frac{1}{z-a}$

$M = \max\limits_{z \in K} |z-a| > 0.$

If $P_n \rightrightarrows f$ in K. find N such that

$$\left| P_N - \frac{1}{z-a} \right| < \frac{1}{2M} \quad \text{in } K$$



$$\Rightarrow \quad |(z-a) P_N - 1| < \frac{|z-a|}{2M} \leq \frac{1}{2}. \quad \text{in } K.$$

$g(z) = 1 - (z-a) P_N$ satisfies

$g(a) = 1$ & $|g(z)| < \frac{1}{2}$ on $\partial H \subseteq K$.

This contradicts maximum modulus for g & the set $\overline{H}$.

Thus f cannot be approximated by polynomials.

Conclusion K has no holes $\iff$ polynomial approximation holds in K.