Math 220 B — Lecture 15

March 6, 2024

Last time

Conway VIII. 1.7.

Thm Let $K \subseteq \mathbb{C}$ compact. Let $S \subseteq \hat{\mathbb{C}} \setminus K$ be a set of points,

at least one chosen from each component of $\hat{\mathbb{C}} \setminus K$.

Let f be holomorphic in K. Then

$\boxed{i}$ $\exists\ R_n \rightrightarrows f$ in K

$\boxed{ii}$ R_n are rational with possible poles in S.

in $\hat{\mathbb{C}}$

Strategy

Step 1 Cauchy Integral Formula for compact sets.

Step 2 Approximation without prescribed poles

Step 3 Push the poles to prescribed location.

Step 1

Recall (Math 220A). $\rightsquigarrow$ Cauchy Integral Formula

$\overline{R} \subseteq \mathcal{U}$ rectangle, then if f holomorphic in $\mathcal{U}$,

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z-a}\, dz = \begin{cases} f(a). & \forall a \in \operatorname{Int} R \\ 0 & \forall a \notin \overline{R} \end{cases}$$

We wish to do the same for any compact $K \subseteq \mathcal{U}$.

not only for rectangles.

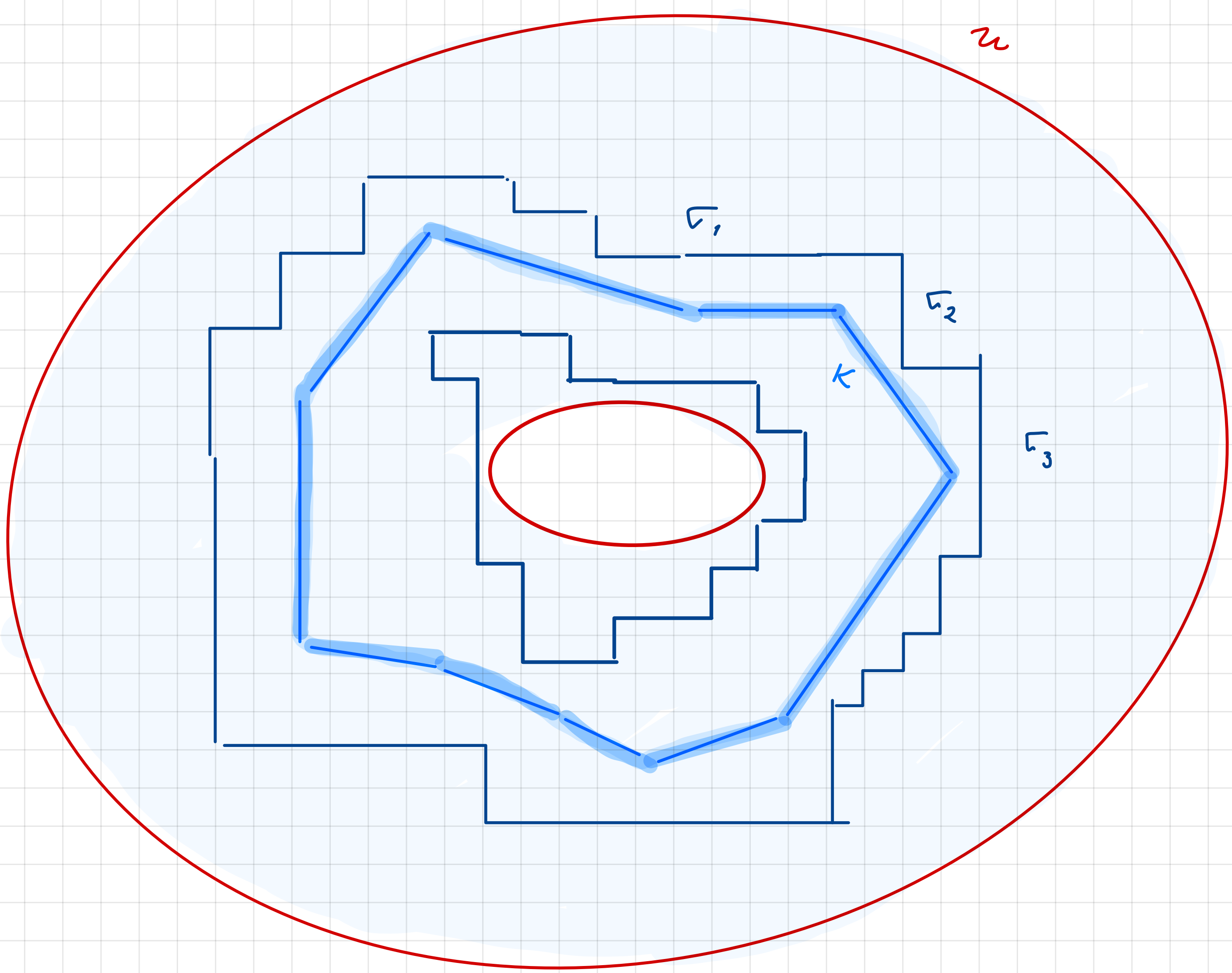
Lemma ← Conway VIII. 1.1.

Let $K \subseteq \mathcal{U}$ compact. There exist segments Γ_j such that

$$\Gamma = \Gamma_1 + \dots + \Gamma_n \subseteq \mathcal{U} \setminus K$$

and such that for all functions f holomorphic in $\mathcal{U}$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz. \qquad \forall\, a \in K.$$



We will construct Γ as a union of closed polygons.

Remark If K has a simple structure this is not so bad. We'd need

$$n(\Gamma, a) = 1 \quad \forall a \in K.$$

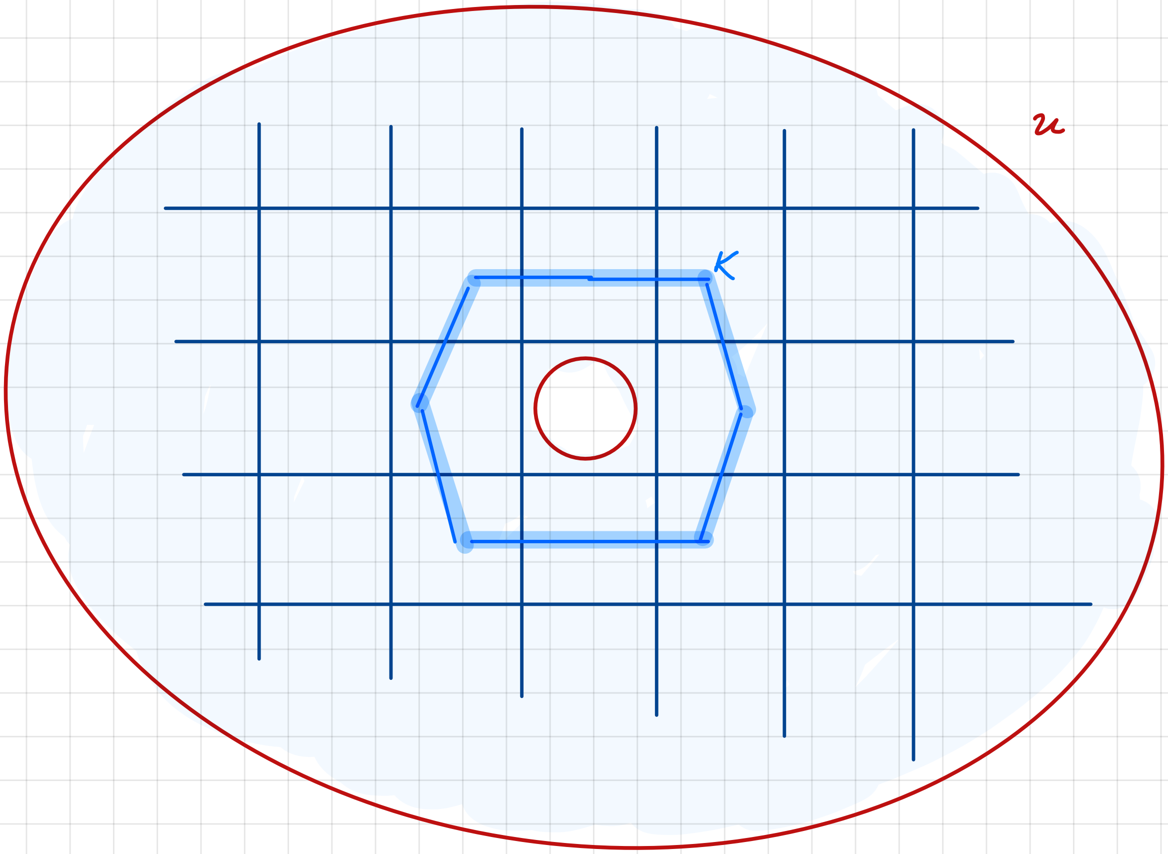
and argue using Cauchy's formula from Math 220A.

The issue is if K has complicated (fractal) structure.

Idea : Lay a grid!

Proof

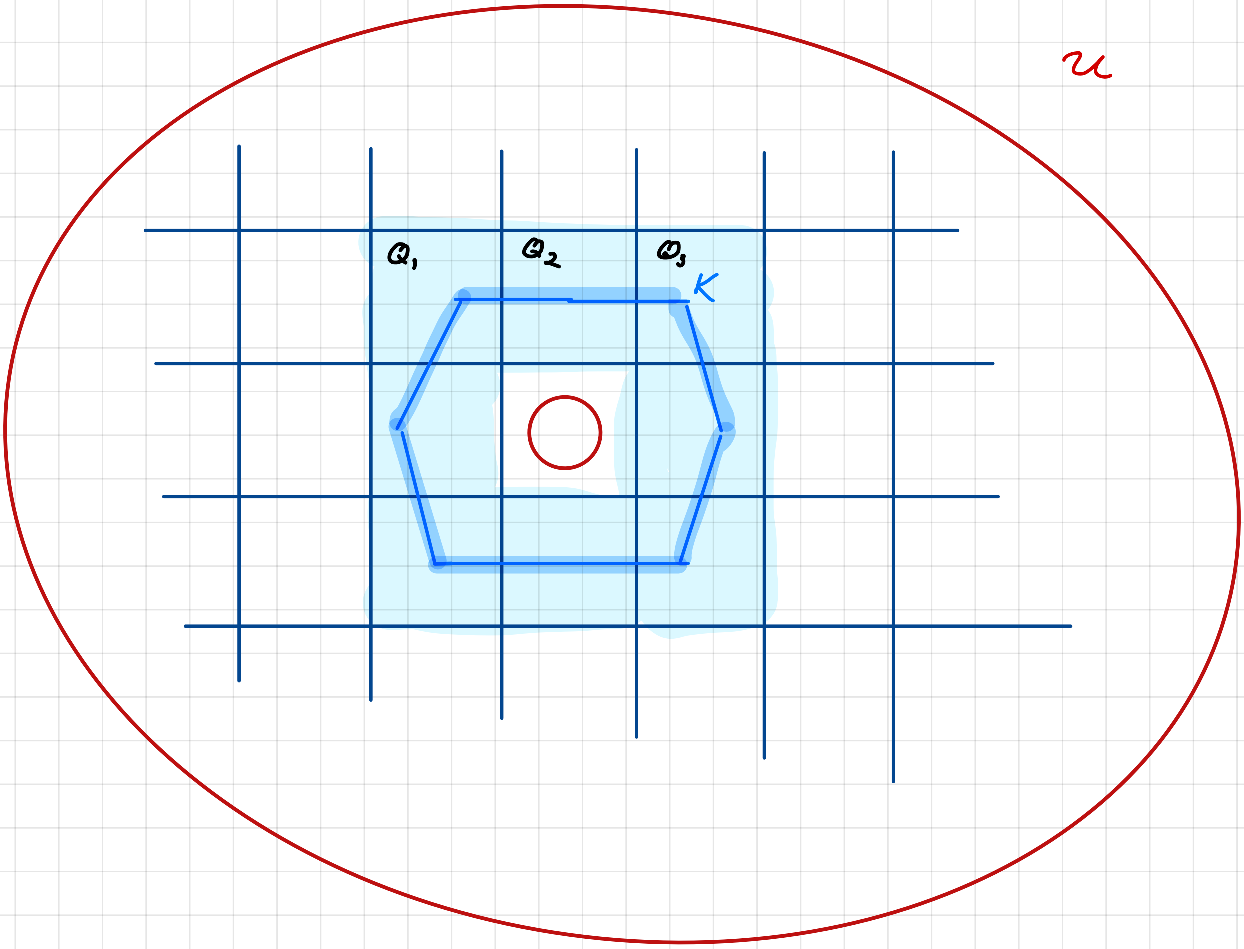
(1) Construction



WLOG $\mathcal{U} \neq \mathbb{C} \Rightarrow \mathbb{C} \setminus \mathcal{U} \neq \emptyset$ is closed. Note

$K \cap (\mathbb{C} \setminus \mathcal{U}) = \emptyset$. Let $d = d(K, \mathbb{C} \setminus \mathcal{U}) > 0$.

Lay a grid of squares of side $< \frac{d}{\sqrt{2}}$.



Consider the closed squares

Q_1, $Q_2 \ldots Q_m$ that intersect K.

There are only finitely many squares since K is compact.

Claim 1 $K \subseteq \bigcup_{j=1}^{m} Q_j \subseteq \mathcal{U}$.

Proof If $k \in K$ then k is contained in a square of the grid. This square intersects K at k so it must be one of the Q_j & $k \in Q_j$. This gives the first inclusion.

For the second inclusion, let $g \in Q_j$ where $Q_j \cap K \neq \emptyset$. Let $k \in Q_j \cap K$. If $g \notin \mathcal{U} \Rightarrow$ $\Rightarrow g \in \mathbb{C} \setminus \mathcal{U}$ and $k \in K$ so

$$d(g, k) \geq d(\mathbb{C} \setminus \mathcal{U}, K) = d.$$

But $g, k \in Q_j \Rightarrow d(g,k) < \operatorname{diam}(Q_j) = d$ contradiction!

Thus $g \in \mathcal{U}$, as needed.

Construction of σ

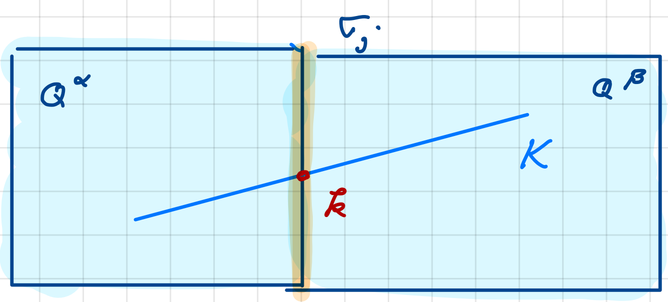
- $\sigma_1, \dots, \sigma_n$ sides of $Q_1, \dots, Q_m$ which are not shared by two squares Q_α, Q_β, $1 \leq \alpha \neq \beta \leq m$.

Claim 2 $\displaystyle\bigcup_{j=1}^{n} \tau_j \subseteq \mathcal{U} \setminus K.$

Proof

Note $\tau_j \subseteq \mathcal{U}$ by Claim 1. Assume $\tau_j \cap K \neq \Phi$.

Let $k \in \tau_j \cap K$. Then τ_j is a side of two squares.



These squares must intersect K necessarily since τ_j does.

These squares must be some of the Q_α, Q_β's,

contradicting the definition of τ_j.

Claim 3 $\forall\, a \in \mathcal{U} \setminus \bigcup_{j=1}^{m} \partial Q_j$ then

$$\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-a}\, dz = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz.$$

This follows because the common sides of the Q_j's cancel out, leaving only the integral over γ_j's.

Assume $a \in \operatorname{Int} Q_\ell$. By Cauchy for rectangles

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-a}\, dz = f(a) \quad \text{if } j = \ell$$

and 0 otherwise.

$\Rightarrow \forall\, a \in \operatorname{Int} Q_\ell$,

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz \qquad (*)$$

This is almost the Lemma. We have one more step.

Claim 4 $\forall\, a \in K$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz \qquad (**).$$

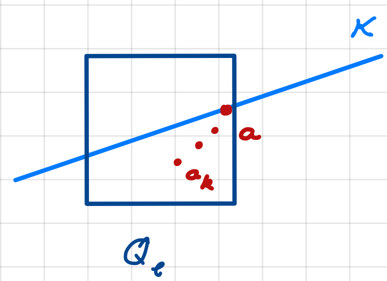
Proof The only issue is the case when $a \notin \bigcup_{j=1}^{m} \operatorname{Int} Q_j$

$\Longrightarrow$ a must be on a side of some Q_ℓ b/c. $K \subseteq \bigcup_{j=1}^{m} Q_j$

by Claim 1. By Claim 2, $a \notin \Gamma_1 + \dots + \Gamma_n$.

Find $a_k \longrightarrow a$ with a_k in the interior of the square Q_ℓ. Both sides of $(**)$ agree at a_k by $(*)$

$$f(a_k) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a_k}\, dz$$



Both sides are continuous in a_k. This is clear for LHS & RHS is explained below. Make $k \to \infty$ to conclude

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz,$$

proving the Lemma completely.

Continuity of RHS is a consequence of:

<u>Key Fact</u> (Math 220A, Homework 3, Problem 5).

$$\Phi : \overset{z}{\Gamma} \times \overset{a}{\mathcal{U} \setminus \Gamma} \longrightarrow \mathbb{C} \quad \text{continuous}$$

then $a \longrightarrow \int_{\Gamma} \Phi(z, a)\, dz$ is continuous.

<u>Apply to</u> $\Phi : \Gamma_j \times \mathcal{U} \setminus \Gamma_j \longrightarrow \mathbb{C}$

$$\Phi(z, a) = \frac{f(z)}{z - a}, \quad z \in \Gamma_j, \quad a \in \mathcal{U} \setminus \Gamma_j.$$

to conclude.

Step 1 is now established. Steps 2 & 3 next time.

Where are we ?

K compact, $K \subseteq \mathcal{U}$, $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic

Wish $\forall\, \varepsilon \quad \exists\, R$ rational function with prescribed poles

in a suitable set S.

$$|f - R| < \varepsilon \quad \text{in } K$$

Conway VIII. 1.1.

Step 1 We found segments $\gamma_1, \dots, \gamma_n \subseteq \mathcal{U} \setminus K$

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(w)}{w - z}\, dw \qquad \forall\, z \in K.$$

Step 2 Find rational functions R with

Conway VIII. 1.5

$|f - R| < \varepsilon$ in K, poles of R are on the segments γ_j.

Step 3 Push the poles to prescribed locations.

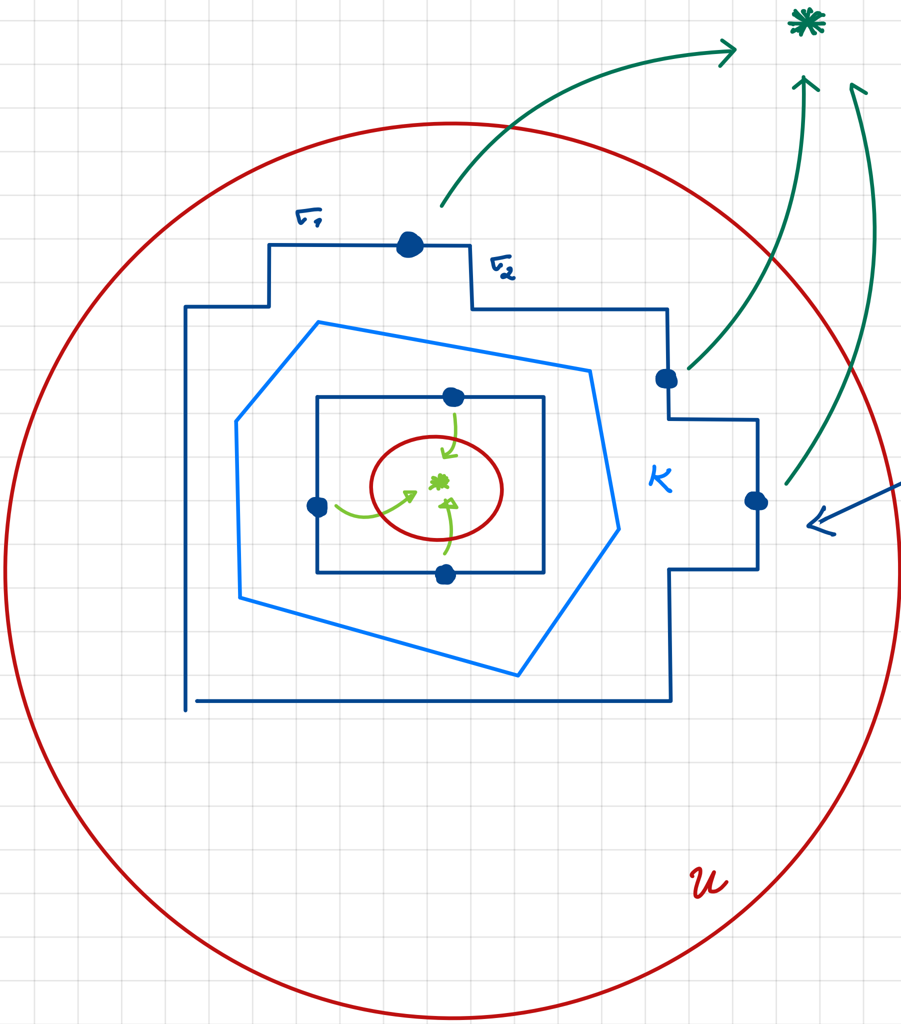
Conway VIII.1.6 – 1.13.

Visualization of the strategy

$S = \{ *, * \}$

↑

prescribed poles



Step 3 will push poles to S

Step 2 will produce poles on Γ_j.

For step 2 we argue one segment Γ_j at a time showing

$$F_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(w)}{w-z}\, dw$$ can be approximated by

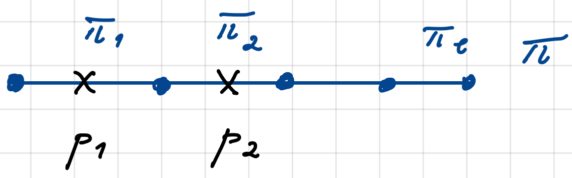
rational functions with poles in Γ_j.

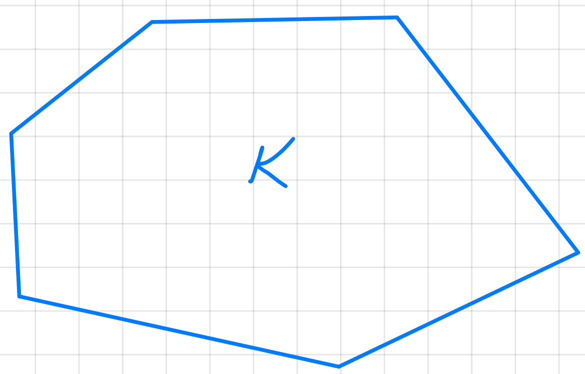
Proof of Step 2

- K compact, π segment (compact), $\pi \cap K = \emptyset$
- f continuous in π

Main Claim (Conway VIII. 1.5)

$F(z) = \int_{\pi} \frac{f(w)}{w-z}\, dw$ can be approximated uniformly on K by rational functions with poles in π.





Proof Let $\varphi(w,z) = \dfrac{f(w)}{w-z} : \pi \times K \longrightarrow \mathbb{C}$, $w \in \pi$, $z \in K$.

Since $\pi \cap K = \emptyset \implies \varphi$ is continuous hence uniformly cont.

$\implies \forall \varepsilon \ \exists \delta$ such that

$$|w - w'| < \delta \implies |\varphi(w,z) - \varphi(w',z)| < \varepsilon.$$

- Subdivide π into subsegments $\pi_1, \dots, \pi_\ell$ of length $< \delta$.
- Pick $p_k \in \pi_k$
- Let $c_k = f(p_k) \displaystyle\int_{\pi_k} dw$
- $R = \displaystyle\sum_{k=1}^{\ell} \frac{c_k}{p_k - z}$.

rational function with pole at $p_k \in \pi$.

Claim

$$\left| F(z) - R(z) \right| = \left| \int_{\pi} \frac{f(w)}{w-z}\, dw - \sum_{k=1}^{\ell} \frac{f(p_k)}{p_k - z} \int_{\pi_k} dw \right|$$

$$= \left| \sum_{k=1}^{\ell} \int_{\pi_k} \left(\frac{f(w)}{w-z} - \frac{f(p_k)}{p_k - z} \right) dw \right|$$

$$\leq \sum_{k=1}^{\ell} \left| \int_{\pi_k} \varphi(w,z) - \varphi(p_k, z)\, dw \right|$$

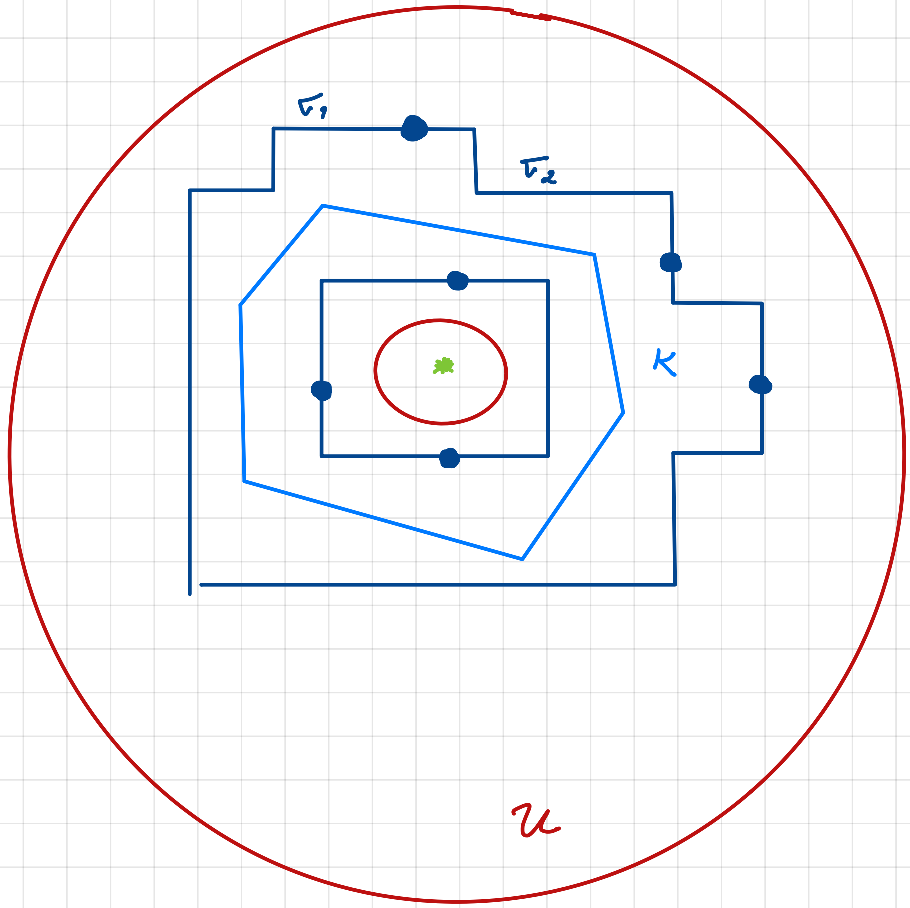
$$\leq \sum_{k=1}^{\ell} \varepsilon \cdot \text{length}(\pi_k) = \varepsilon \cdot \text{length}(\pi).$$

Here we used $|\varphi(w,z) - \varphi(p_k, z)| < \varepsilon$ since $|w - p_k| < \delta$ which is true as $p_k, w \in \pi_k$, $\text{length}(\pi_k) < \delta$.

The proof of Step 2 is completed.

● → poles in Step 2

*, * → prescribed poles



Where are we?

- $K \subseteq \mathcal{U}$, f holomorphic
- $\exists\, R$ with poles in Γ_j, $|f - R| < \varepsilon$ on K.

Final Step Fix S a set of poles, one from each component of $\hat{\mathbb{C}} \setminus K$.

Push the poles from Γ_j to the points of S.

Step 3   Pole pushing to prescribed location.

Let $\hat{\mathbb{C}} \setminus K = \bigcup_i H_i$ = connected components

Let H be a fixed component.

pole produced in Step 2

prescribed location

Lemma $\forall\ a, b \in H$. Then

$\frac{1}{z-a}$ can be approximated uniformly in K by

poles at b

polynomials in $\frac{1}{z-b}$

If H is unbounded & $b = \infty$ then

$\frac{1}{z-a}$ can be approximated uniformly in K by polynomials

Polynomials in z = Rational Functions with poles possibly only at ∞.

Proof of the Lemma

- Keep b fixed & vary a. Consider the set

- $W = \Big\{ c \in H : \dfrac{1}{z-c}$ can be approximated uniformly in K

  polynomials in $\dfrac{1}{z-b} \Big\}$

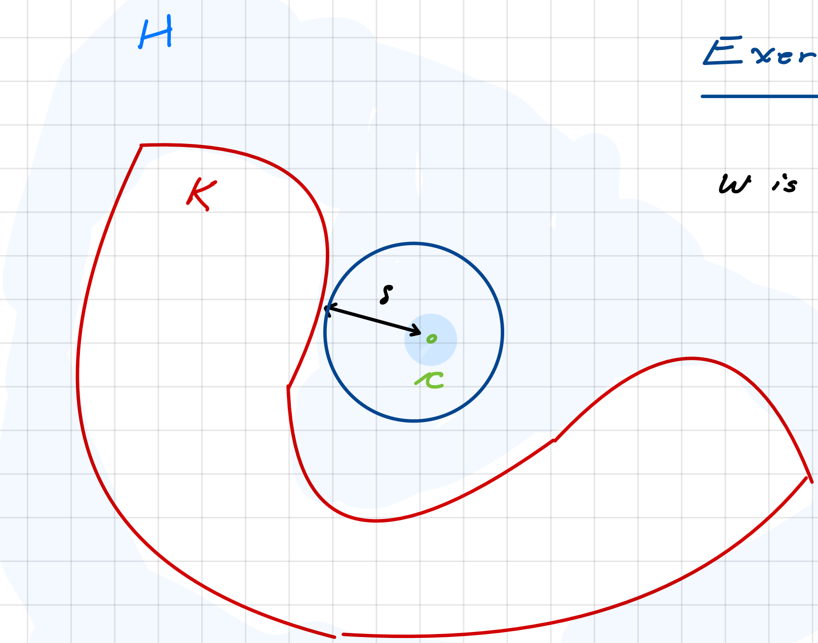
We wish to prove $W = H$.

- $W \neq \Phi$ because $b \in W$.

Key Claim

(*) $\forall\ c \in W$, let $\delta = d(c, K)$. Then $\Delta(c, \delta) \subseteq W$.

$\Downarrow$

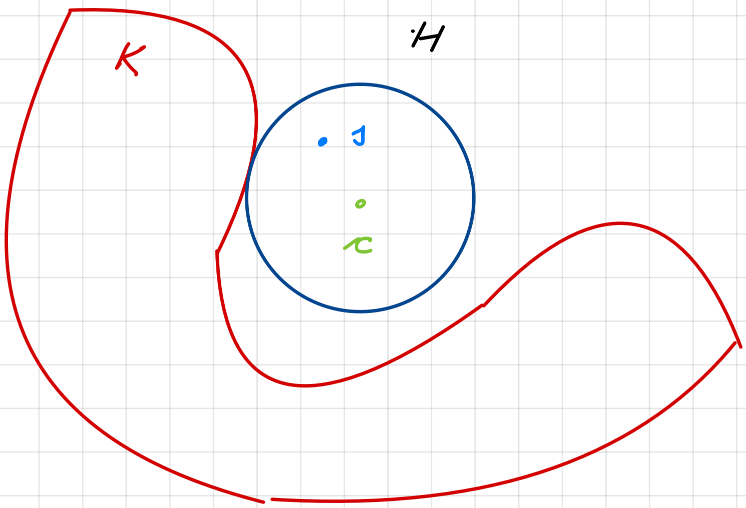


Exercise This implies

W is closed & open hence $W = H$.

Proof of Key Claim

Let $s \in \Delta(c, \delta)$. We wish to show that $s \in W$. $\Rightarrow \Delta \subseteq W$ as needed.



Idea $\frac{1}{z-s} \rightsquigarrow$ poly in $\frac{1}{z-c} \rightsquigarrow$ poly in $\frac{1}{z-b}$. $\Rightarrow s \in W$.

Consider the Laurent expansion of $\frac{1}{z-s}$ at c in $\Delta(c; \delta, \infty)$

$$\frac{1}{z-s} = \frac{1}{z-c} \cdot \frac{1}{1 - \frac{s-c}{z-c}} = \frac{1}{z-c} \sum_{k \geq 0} \left(\frac{s-c}{z-c}\right)^k = \sum_{k \geq 0} \frac{(s-c)^k}{(z-c)^{k+1}}$$

Convergence: $|z-c| > \delta > |s-c|$.

Note $z \in K$, $\delta = d(c, K) \Rightarrow K \subseteq \Delta(c; \delta, \infty)$. The Laurent expansion in $\Delta(c; \delta, \infty)$ converges locally uniformly

(Math 220 A, Lecture 9.)

Pick T a Laurent polynomial in $\frac{1}{z-c}$ from the Laurent expansion above so that

$$\left| \frac{1}{z-s} - T \right| < \frac{\varepsilon}{2} \quad \text{over } K.$$

Since $c \in W \Rightarrow \frac{1}{z-c}$ can be approximated by polynomials in $\frac{1}{z-b}$. The same is then true about $T =$ polynomial in $\frac{1}{z-c}$. Then $\exists\, P$ polynomial in $\frac{1}{z-b}$ so that

$$|T - P| < \frac{\varepsilon}{2} \quad \text{in } K$$

Then $\left| \frac{1}{z-s} - P \right| \le \left| \frac{1}{z-s} - T \right| + |T - P| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ in K. This

polynomial in $\frac{1}{z-b}$.

shows $s \in W$.

If H is unbounded

Let $K \subseteq \Delta(0,r)$

- first move the poles to $|c| > r$.

- Taylor expand $\frac{1}{z-c}$ near $z=0$ in

$$\Delta(0,|c|) \supseteq \Delta(0,r) \supseteq K$$

The Taylor series converges locally uniformly. Hence we can approximate $\frac{1}{z-c}$ by polynomials uniformly on K.

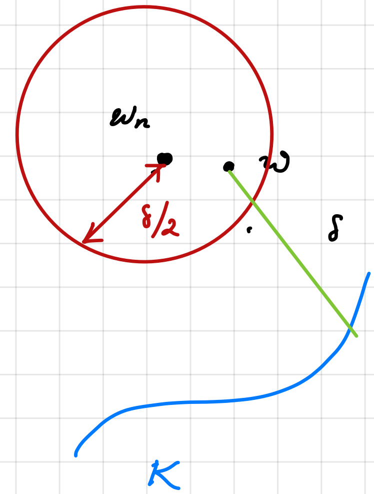
Proof of the Exercise

- W open. Indeed $\forall c \in W \;\exists\, \Delta(c,\delta) \subseteq W$

by $(*)$ showing W open

- We show W closed in H.

Assume $w_n \longrightarrow w$, $w_n \in W$, $w \in H$.



Let $d(w, K) = \delta > 0$ since $w \notin K$.

Fix n with $d(w, w_n) < \frac{\delta}{2}$.

$$\Longrightarrow \quad d(w_n, K) \geq d(w, K) - d(w, w_n) > \frac{\delta}{2}$$

$$\Longrightarrow \quad \Delta\left(w_n, \frac{\delta}{2}\right) \subseteq W \quad \text{since } w_n \in W \text{ and } (*)$$

$w \in$

$\Longrightarrow$ $w \in W$. since $w \in \Delta\left(w_n, \frac{\delta}{2}\right)$. This proves the

Exercise.

Remark This completes the proof of Runge.

Summary: start with f $\overset{\text{Step 1}}{\rightsquigarrow}$ Cauchy for compact sets

$\overset{\text{Step 2}}{\rightsquigarrow}$ rational approximation with poles in Γ_j.

$\overset{\text{Step 3}}{\rightsquigarrow}$ further approximation with prescribed poles