Math 220B — Lecture 16

March 11, 2024

§0. Last time — We established Runge C

Thm [i] $K \subseteq \mathbb{C}$ compact, $S \subseteq \hat{\mathbb{C}} \setminus K$ contains a point from each component of $\hat{\mathbb{C}} \setminus K$.

[ii] f holomorphic in K

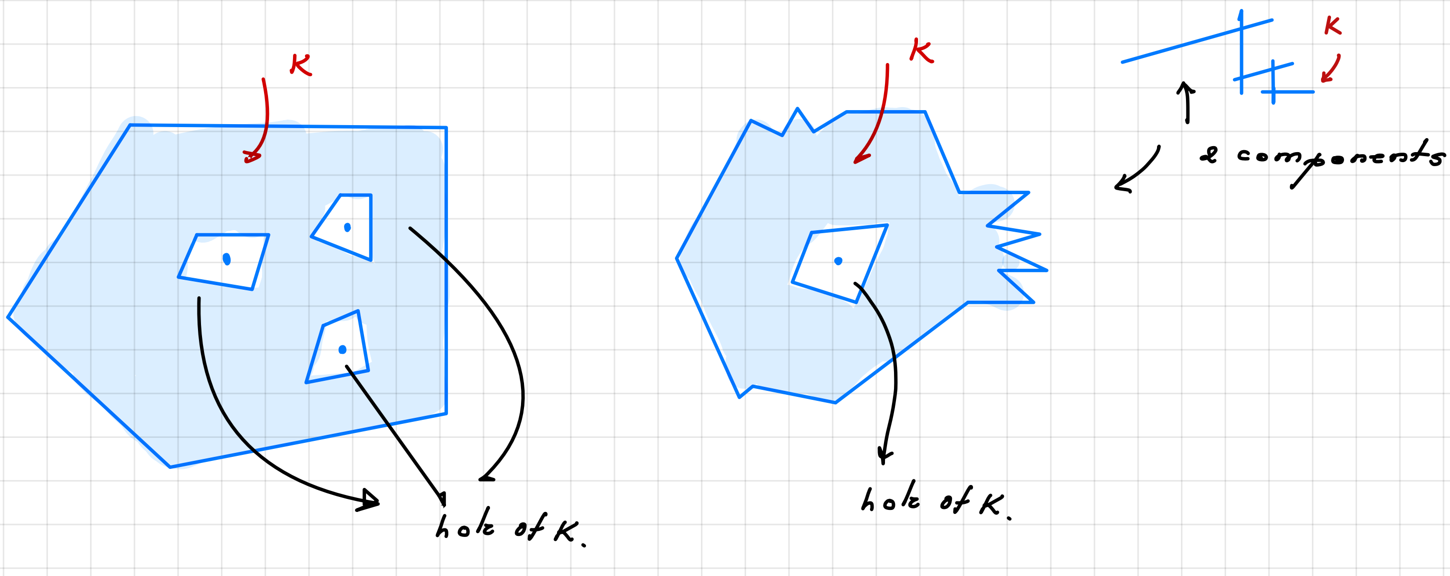
$\Rightarrow$ $\forall\, \varepsilon$ $\exists\, R$ rational,

$|f - R| < \varepsilon$ in K and $\text{poles}(R) \subseteq S$.

Remark For $\varepsilon = \frac{1}{n} \Rightarrow \exists\, R_n$ with $|f - R_n| < \frac{1}{n}$ in K

$\Rightarrow R_n \rightrightarrows f$ in K. & $\text{poles}(R_n) \subseteq S$.

The set K can be disconnected and quite strange.



Applications

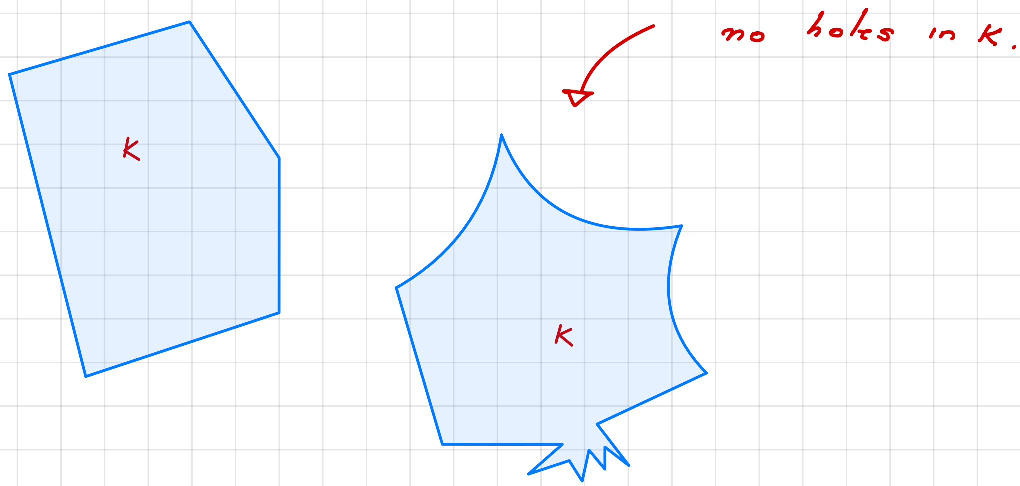
- density in spaces of functions
- new proof of Mittag-Leffler Conway VIII. 3.
- polynomial convexity Conway VIII. 1.
- generalizations: Mergelyan, ...

Important Special Case - Little Runge C

K has no holes $\Rightarrow$ $\hat{\mathbb{C}} \setminus K$ has only one unbounded component & we can take $S = \{\infty\}$

$\Downarrow$

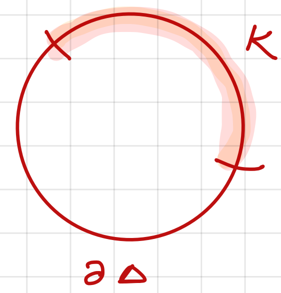
All f holomorphic in K can be approximated uniformly in K by polynomials.



The set K can be disconnected

Example Let K be an arc of $\partial\Delta$, $K \neq \partial\Delta$.

Then $\exists\, P$ polynomial, $P(0) = 1$, $|P|_K < 1$.



Proof

Indeed, $\mathbb{C} \setminus K$ connected $\Rightarrow$ polynomial approximation holds on K. The function $f(z) = \frac{1}{z}$ is holomorphic in K.

Thus $\exists$ polynomial Q with

$$\left| Q - \frac{1}{z} \right| < 1 \quad \text{on } K.$$

$\Rightarrow$ $|1 - zQ| < |z| = 1$ on K. Set $P(z) = 1 - z\,Q(z)$ and note $P(0) = 1$ & $|P| < 1$ on K.

Remark The statement is false if $K = \partial\Delta$. Indeed, if P existed, we'd contradict maximum modulus principle.

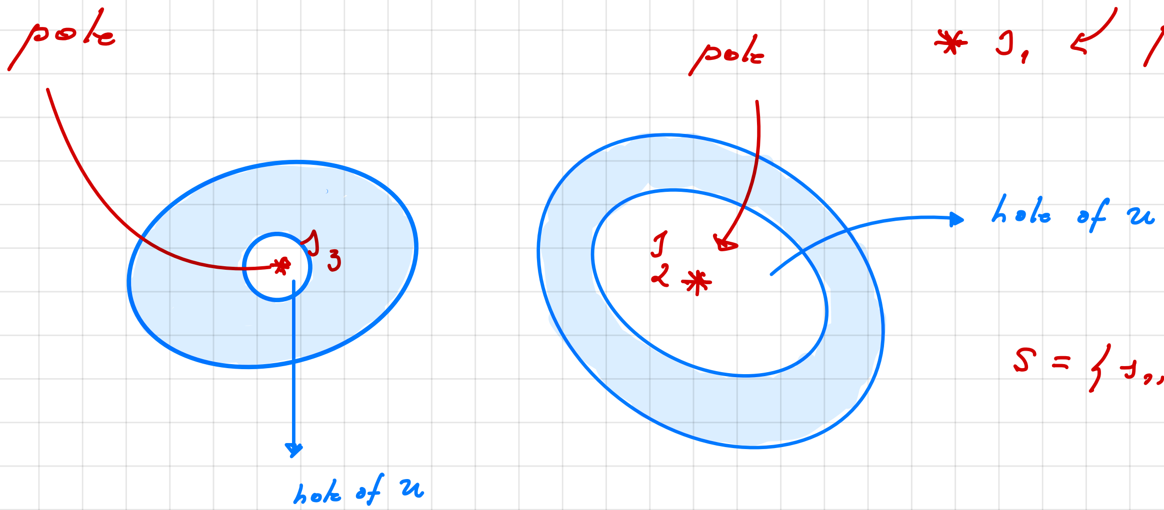
§2. Runge for Open Sets ← Conway VIII. 1.15.

- We approximate locally uniformly on open sets
- the statement is similar to Runge for compact sets

Theorem
- $\mathcal{U} \subseteq \mathbb{C}$ possibly disconnected. open set.
- $S \subseteq \hat{\mathbb{C}} \setminus \mathcal{U}$ containing at least a point from each component of $\hat{\mathbb{C}} \setminus \mathcal{U}$.
- $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

Then $\exists\, R_n$ rational functions, poles$(R_n) \subseteq S$ and

$$R_n \overset{\ell.u.}{\rightrightarrows} f \quad \text{locally uniformly in } \mathcal{U}.$$



$* \; s_1 \leftarrow$ pole

$S = \{s_1, s_2, s_3\}$

Important Special Case (Little Runge 0)

Let $\mathcal{U} \subseteq \mathbb{C}$, open, $\hat{\mathbb{C}} \setminus \mathcal{U}$ connected.

Any $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic can be approximated locally uniformly on $\mathcal{U}$ by polynomials.

Indeed, take $S = \{\infty\}$ in Runge 0., R_n's can't have any denominators since those will yield poles of R_n which are not in S.

Example Let $\mathcal{U} = \Delta(0,r)$, $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

We can Taylor expand f in the disc. The Taylor polynomials

$$T_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k \quad \& \quad T_n \overset{\text{l.u.}}{\rightrightarrows} f \quad (\text{Math 220A}).$$

Little Runge 0 applies to more general sets $\mathcal{U}$.

Proof of Runge Open

Conway VII.1.2.

Topological Lemma For $\mathcal{U} \subseteq \mathbb{C}$ open, we can find $\underbrace{K_n}_{\text{compact}} \subseteq \mathcal{U}$

$$(*) \qquad \mathcal{U} = \bigcup_{n \geq 1} K_n$$

exhausting compact sets

[i] $K_n \subseteq \operatorname{Int} K_{n+1}$

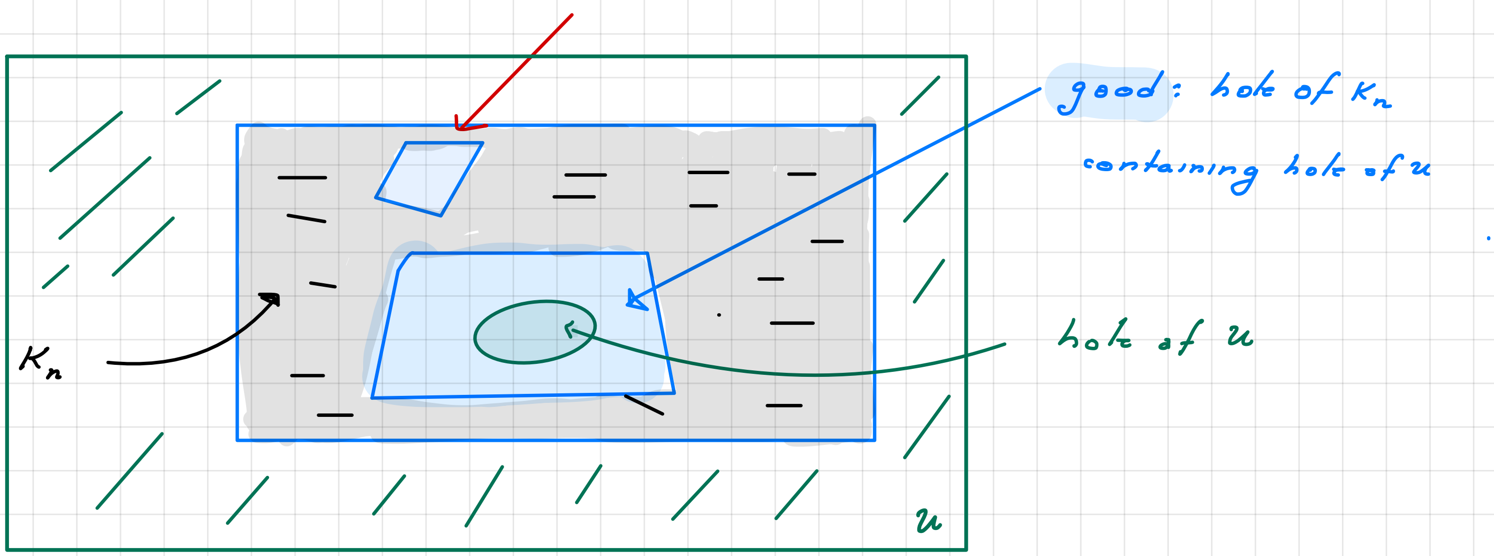
[ii] $\forall\, K \subseteq \mathcal{U}$ compact $\Rightarrow \exists\, n,\ K \subseteq K_n$.

[iii] each component of $\hat{\mathbb{C}} \setminus K_n$ contains a component of $\hat{\mathbb{C}} \setminus \mathcal{U}$.

Remark [iii] $\Rightarrow$ holes of K_n contain holes of $\mathcal{U}$.

Good vs. bad

bad: (hole of K_n, but not of $\mathcal{U}$. We don't want that!



Topological Lemma $\Rightarrow$ Runge O

Let $f: \mathcal{U} \to \mathbb{C}$ holomorphic. Let S contain a point from each component of $\hat{\mathbb{C}} \setminus \mathcal{U}$. Write

$$\mathcal{U} = \bigcup_{n \geq 1} K_n \quad \text{as in the Lemma.}$$

The set S contains a point from each component of $\hat{\mathbb{C}} \setminus K_n$ by $\boxed{iii}$. By Runge C applied to f & K_n, we find

$$|f - R_n| < \frac{1}{n} \quad \text{in } K_n, \quad \text{poles}(R_n) \subseteq S.$$

We claim $R_n \overset{\text{l.u.}}{\rightrightarrows} f$. Let K be compact in $\mathcal{U}$. By $\boxed{ii}$

$\Rightarrow K \subseteq K_N$ for some N. For $n \geq N \Rightarrow K \subseteq K_N \subseteq K_n$ by $\boxed{i}$

$\Rightarrow |f - R_n| < \frac{1}{n}$ over $K_n \Rightarrow |f - R_n| < \frac{1}{n}$ in K.

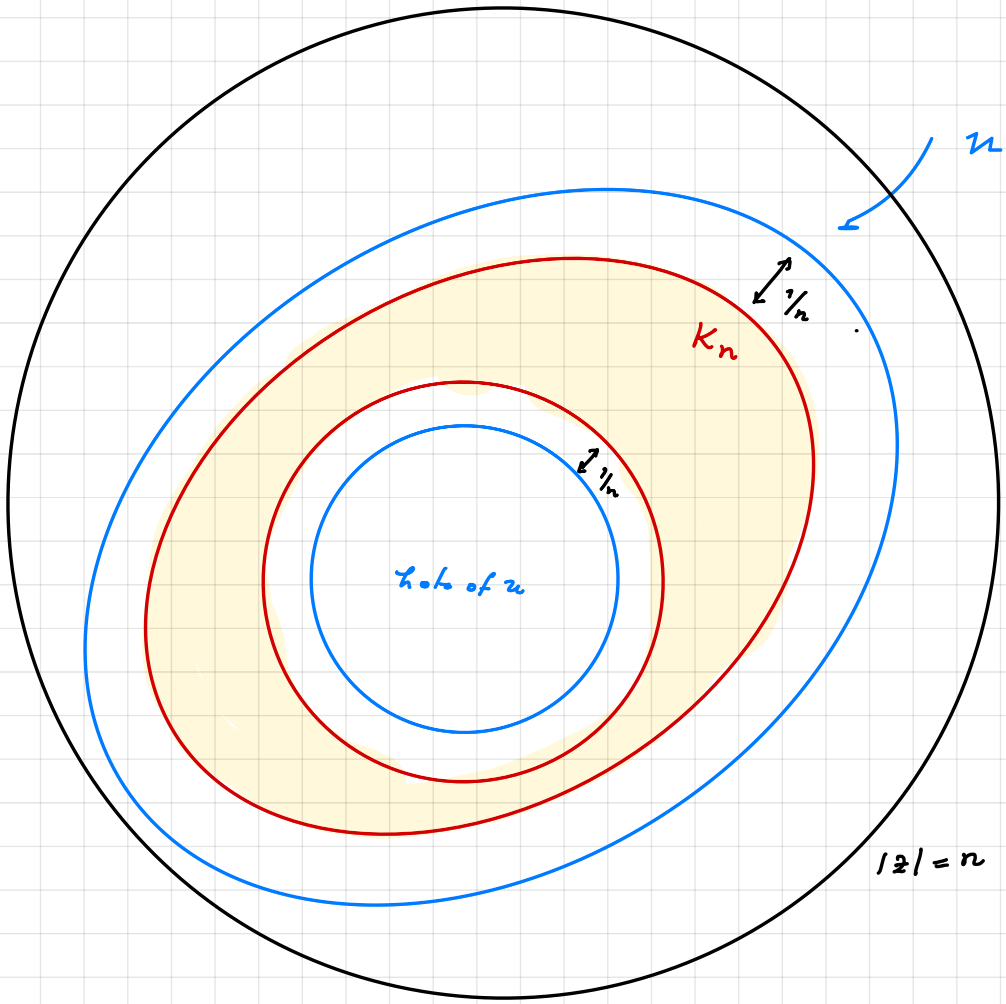
Thus $R_n \rightrightarrows f$ in K, as needed.

Proof of the Topological Lemma

} Conway VII.1.2

WLOG $\mathcal{U} \neq \mathbb{C}$.

Let $K_n = \{ z : |z| \le n \text{ and } d(z, \underbrace{\mathbb{C} \setminus \mathcal{U}}_{\text{closed.}}) \ge \frac{1}{n} \}$.



It is easy to see $\boxed{\text{I}}$ – $\boxed{\text{III}}$ hold, using the above pictures.

The technical details follow (see also Conway).

$$K_n = \{ z : |z| \le n \text{ and } d(z, \mathbb{C} \setminus \mathcal{U}) \ge \tfrac{1}{n} \}.$$

Claim 1 $K_n \subseteq \mathcal{U}$

Proof If $z \in K_n \Rightarrow d(z, \mathbb{C} \setminus \mathcal{U}) \ge \frac{1}{n} \Rightarrow z \notin \mathbb{C} \setminus \mathcal{U} \Rightarrow z \in \mathcal{U}$. Thus $K_n \subseteq \mathcal{U}$.

Claim 2. $\mathcal{U} = \bigcup_{n \ge 1} K_n$

Proof If $z \in \mathcal{U}$ then let n such that $n \ge |z|$ & $d(z, \mathbb{C} \setminus \mathcal{U}) \ge \frac{1}{n}$ which is possible since $d(z, \mathbb{C} \setminus \mathcal{U}) > 0$. Thus $z \in K_n$. $\Rightarrow \mathcal{U} \subseteq \bigcup_n K_n \overset{\text{claim 1}}{\subseteq} \mathcal{U}$

Claim 3 K_n closed & bounded $\Rightarrow$ K_n compact.

Proof K_n is closed since

$$\mathbb{C} \setminus K_n = \{ |z| > n \} \cup \{ z : \exists\, b \notin \mathcal{U},\ d(z, b) < \tfrac{1}{n} \}$$

$$= \{ |z| > n \} \cup \bigcup_{b \notin \mathcal{U}} \Delta(b, \tfrac{1}{n}). = \text{open}.$$

Claim 4 $K_n \subseteq \operatorname{Int} K_{n+1}$

Proof Let $z \in K_n$. Let $r < \frac{1}{n} - \frac{1}{n+1}$. then

$$\Delta(z, r) \subseteq K_{n+1} \implies z \in \operatorname{Int} K_{n+1} \quad \text{as needed.}$$

To see $\Delta(z,r) \subseteq K_{n+1}$ note for $w \in \Delta(z,r)$

$$|w| \leq |z| + |w - z| \leq n + r < n+1 \quad \text{and}$$

$$d(w, \mathbb{C} \setminus \mathcal{U}) \geq d(z, \mathbb{C} \setminus \mathcal{U}) - d(z, w) \geq \frac{1}{n} - r > \frac{1}{n+1}.$$

$\implies w \in K_{n+1}$, as needed.

Claim 5 Each compact $K \subseteq \mathcal{U}$ is contained in some K_n.

Proof Let $K \subseteq \mathcal{U} \overset{\text{claim 2}}{=} \bigcup_m K_m \overset{\text{claim 4}}{\subseteq} \bigcup_m \operatorname{Int} K_{m+1}$. Since K is compact we find a finite subcover by $\operatorname{Int} K_j$, $j \leq n$.

$$\implies K \subseteq \bigcup_{j \leq n} \operatorname{Int} K_j \underset{\text{claim 4.}}{\subseteq} K_n$$

Claim 6 Let $A = \hat{\mathbb{C}} \setminus K_n$, $B = \hat{\mathbb{C}} \setminus \mathcal{U}$ $\Rightarrow A \supseteq B \ni \infty$

(+) Each component of A contains a component of B.

Proof This is a bit more technical. We will use repeatedly:

Easy important fact (by definition)

If $Z \subseteq A$ connected & Z intersects a component A° of A

$\Rightarrow Z \subseteq A^\circ$.

Proof of (+) Let A° be a component of A. By Claim 3 (proof):

$$A = \underbrace{\{z \in \hat{\mathbb{C}} : |z| > n\}}_{\text{contains } \infty} \cup \bigcup_{b \in B} \Delta\left(b, \tfrac{1}{n}\right)$$

[I] Note $\infty \in A$. If A° is the component containing ∞, let B° be the component of B containing $\infty \in B$. Note

$$A^\circ \cap B^\circ \neq \emptyset \text{ (contains } \infty) \;\&\; B^\circ \subseteq A \underset{\text{easy fact}}{\Longrightarrow} B^\circ \subseteq A^\circ.$$

This is what we wanted to show.

$\boxed{ii}$ If $\infty \notin A^\circ$, then A° cannot be disjoint from all sets $\Delta(b, \frac{1}{n})$.

Why Else $A^\circ \subseteq \underbrace{\Delta(\infty, n)}_{\text{connected set \& intersects } A^\circ} \subseteq A \underbrace{\Rightarrow}_{\text{easy fact.}} \Delta(\infty, n) \subseteq A^\circ \Rightarrow \infty \in A^\circ$.

false.

Thus $\exists\, b \in B$ with $A^\circ \cap \Delta(b, \frac{1}{n}) \neq \Phi$. Note

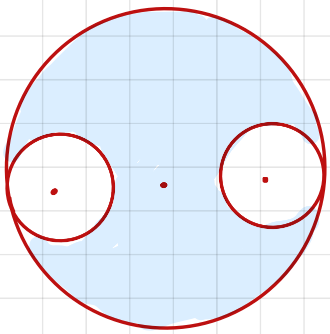
$$\Delta(b, \tfrac{1}{n}) \subseteq A \text{ \& intersects } A^\circ \underset{\text{easy fact}}{\Longrightarrow} \Delta(b, \tfrac{1}{n}) \subseteq A^\circ.$$

Let $b \in B^\circ$ for some component B°.

Then $B^\circ \cap A^\circ \neq \Phi$ & $B^\circ \subseteq B \subseteq A \underset{\text{easy fact}}{\Longrightarrow} B^\circ \subseteq A^\circ$ as needed.

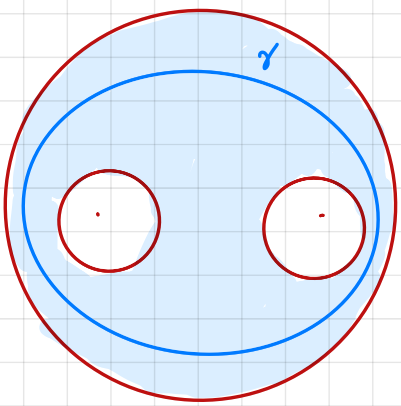
Example (Little Runge 0)

$$U = \{ z : |z| < 5, \ |z-4| > 1, \ |z+4| > 1 \}.$$



- $\mathbb{C} \setminus U$ connected $\Rightarrow$ polynomial approx. holds in U.

If instead we take $U = \{ z : |z| < 5, \ |z-3| > 1, \ |z+3| > 1 \}$.



Polynomial approximation fails.

Indeed, let $f(z) = \dfrac{1}{z-3}$,

$\gamma = \{ |z| = 4\tfrac{1}{2} \}$. If $p_n \overset{\text{l.u.}}{\rightrightarrows} f$ on U

then $\underbrace{\int_\gamma p_n \, dz}_{0} \longrightarrow \underbrace{\int_\gamma f \, dz}_{2\pi i}$. false!