Math 220B — Lecture 17

March 13, 2024

Putting the pieces together   Conway VIII.2.

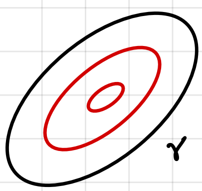
We tie up loose ends from Math 220A & B

Common theme : simply connected regions.

Topology $\longleftrightarrow$ Analysis

Review of Lecture 11, 220A $\mathcal{U} \subseteq \mathbb{C}$ connected

$\boxed{\text{I}}$ $\mathcal{U}$ is simply connected iff $\forall \gamma$ closed path in $\mathcal{U}$



$$\gamma \overset{\mathcal{U}}{\sim} 0$$

$\boxed{\text{II}}$ γ piecewise C^1 loop in $\mathcal{U}$, $\gamma \overset{\mathcal{U}}{\approx} 0$ (null homologous) iff

$$\forall\ a \notin \mathcal{U}, \quad n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 0$$

Recall

$$\gamma \overset{U}{\sim} 0 \implies \gamma \overset{U}{\approx} 0$$

Indeed, $a \notin U$.

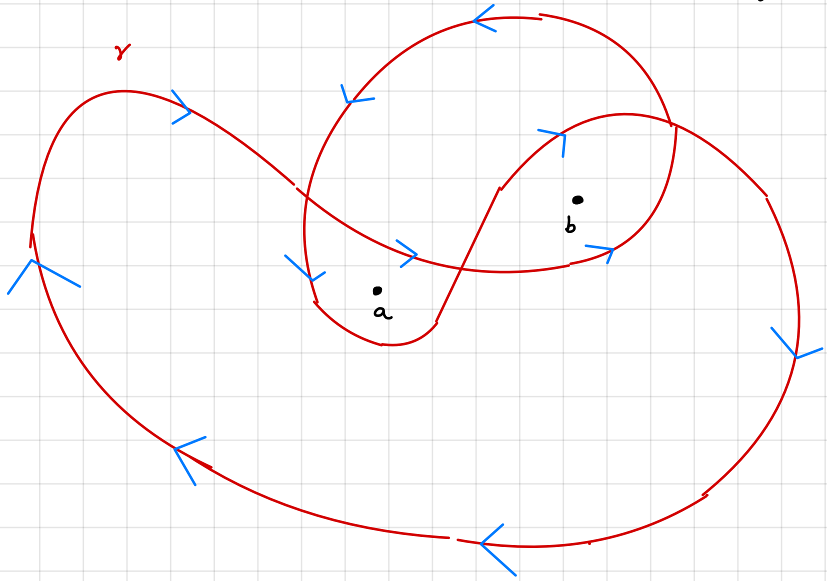
$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0$ by the homotopy form of Cauchy applied to the holomorphic function

$z \longrightarrow \frac{1}{z-a}$ in U.

However

the converse is false $U = \mathbb{C} \setminus \{a, b\}$

$\gamma \not\sim 0$, $\gamma \approx 0$.



Theorem Let $U \subseteq \mathbb{C}$ open, connected. TFAE

Conway VIII.2.

[a] U simply connected

$\Downarrow$

[b] $\forall \gamma$ piecewise C^1 loop, $\gamma \overset{U}{\approx} 0$

$\Downarrow$

[c] $\hat{\mathbb{C}} \setminus U$ connected.

$\Downarrow$

[d] polynomial approximation $\forall f$ holomorphic in U can be approximated $p_n \overset{\text{l.u.}}{\rightrightarrows} f$ in U

$\Downarrow$

[e] $\forall \gamma$ piecewise C^1 loop, f holomorphic in U

$$\int_\gamma f \, dz = 0.$$

$\Downarrow$

[f] primitives: any holomorphic $f : U \to \mathbb{C}$ admits a primitive.

$\Downarrow$

[g] logarithms: $\forall f : U \to \mathbb{C}$ holomorphic, nowhere zero can be written $f = e^g$, $g : U \to \mathbb{C}$ holomorphic.

$\Downarrow$

[h] roots: $\forall f : U \to \mathbb{C}$ holomorphic, nowhere zero can be written $f = h^2$, $h : U \to \mathbb{C}$ holomorphic.

$\Downarrow$

[i] U is homeomorphic to $\Delta(0,1)$.

Recall $\quad \mathcal{U}, V \subseteq \mathbb{C}$ are homeomorphic if $\exists\, f : U \longrightarrow V$

$g : V \longrightarrow \mathcal{U}$ continuous & inverse to each other.

Proof

$\boxed{a} \Longrightarrow \boxed{b}$ This is the statement $\gamma \overset{U}{\sim} 0 \Rightarrow \gamma \overset{U}{\approx} 0$.

that we saw previously.

$\boxed{b} \Longrightarrow \boxed{c}$ Assume $\hat{\mathbb{C}} \setminus U = A \cup B$

$A, B \neq \emptyset$ closed & disjoint. Assume $\infty \in B$. $\Longrightarrow$

$\Longrightarrow$ A is closed in $\underbrace{\hat{\mathbb{C}} \setminus U}_{\text{closed in } \hat{\mathbb{C}}} \Rightarrow A$ closed in $\hat{\mathbb{C}} \Rightarrow A$ compact.

Let $\underset{\not\ni \infty}{V} = U \cup A = \hat{\mathbb{C}} \setminus \underset{\text{closed}}{B}$. $\Rightarrow V$ open subset of $\mathbb{C}$, $A \subseteq V$.

In Lecture 15, we saw Cauchy's formula for compact sets.

A = compact, $A \subseteq V$. $\Rightarrow \exists$ polygons $\Gamma_1 \dots \Gamma_n$ in $V \setminus A = U$.

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz \quad \forall\, a \in A,\ f \text{ holom. in } V.$$

Take $f \equiv 1$ then $1 = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{dz}{z-a} = \sum_{j=1}^{n} \underbrace{n(\Gamma_j, a)}_{0}$.

However, by assumption $n(\Gamma_j, a) = 0 \ \forall j$ since Γ_j is a piecewise C^1 loop in U and $a \in A \Rightarrow a \notin U$. This contradicts

$$\sum_{j=1}^{n} n(\Gamma_j, a) = 1.$$

$\boxed{c} \Rightarrow \boxed{d}$ this is Little Runge 0.

$\boxed{d} \Longrightarrow \boxed{e}$ If $p_n \overset{l.u.}{\rightrightarrows} f$ in $\mathcal{U}$ then $\int_\gamma p_n \, dz \longrightarrow \int_\gamma f \, dz$.

However p_n admits a primitive $p_n = g_n'$ so by

Lecture 5, Math 220 A $\quad \int_\gamma p_n \, dz = \int_\gamma g_n' \, dz = 0$

$$\Longrightarrow \int_\gamma f \, dz = 0.$$

$\boxed{e} \Longrightarrow \boxed{f}$ This was done in Lecture 4, Math 220 A

$\boxed{f} \Longrightarrow \boxed{g}$ Math 220 A, Homework 4. Recall the argument.

Consider $\frac{f'}{f}$ holomorphic in $\mathcal{U}$. Then $\frac{f'}{f} = g'$ for some g by $\boxed{f}$

$$\Longrightarrow (e^{-g} f)' = 0 \Longrightarrow f = c e^{g} = e^{\tilde{g}}, \quad \tilde{g} = g + \log c, \quad c \neq 0.$$

$\boxed{g} \Longrightarrow \boxed{h}$ Write $f = e^{g}$ and let $h = e^{g/2}$.

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[h] $\Rightarrow$ [i] If $\mathcal{U} \neq \mathbb{C}$, Riemann Mapping shows $\mathcal{U}$ and Δ are biholomorphic hence homeomorphic.

If $\mathcal{U} = \mathbb{C}$ then $z \longrightarrow \frac{z}{\sqrt{1+|z|^2}}$ is a homeomorphism between $\mathbb{C}$ and Δ.

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[i] $\Rightarrow$ [a] Let f, g be the two inverse homeomorphisms $\mathcal{U} \underset{g}{\overset{f}{\rightleftarrows}} \Delta$.

Let γ be a loop in $\mathcal{U}$ $\Rightarrow f \circ \gamma \overset{\Delta}{\sim} 0 \Rightarrow g \circ f \circ \gamma \overset{\mathcal{U}}{\sim} g(0). \Rightarrow \gamma \overset{\mathcal{U}}{\sim} g(0)$

$\Rightarrow \mathcal{U}$ simply connected.

Remark The implications $a \Rightarrow b, c, d, e \ldots$ are very useful.

For the converse, $c \Rightarrow a$ is important.

Remark

Topology: $a, c, i \ldots$

Analysis: $d, e, f, g, \ldots$

We can add one more statement to the list.

Proposition $\mathcal{U} \subseteq \mathbb{C}$ open

$\boxed{a}$ $\mathcal{U}$ simply connected

$\boxed{b}$ every harmonic function $u : \mathcal{U} \to \mathbb{R}$ admits a

class C^2

harmonic conjugate v (i.e. $u + iv$ holomorphic in $\mathcal{U}$).

Proof $\boxed{b} \Rightarrow \boxed{a}$ We show $\mathcal{U}$ is a logarithm domain.

This implies $\mathcal{U}$ simply connected by the theorem above.

Let $f : \mathcal{U} \to \mathbb{C}$ be nowhere zero. We show $\exists\, g : \mathcal{U} \to \mathbb{C}$ holomorphic with $f = e^g$. Let $h = \log|f|$. Then h is harmonic. (This can be checked directly using Cauchy-Riemann equations $\rightsquigarrow$ Exercise)

Let v be the harmonic conjugate of h.

$\Rightarrow g = h + iv$ is holomorphic & $|e^g| = |e^{h+iv}| = e^h = |f| \Rightarrow$

$\Rightarrow |f e^{-g}| = 1 \Rightarrow f e^{-g} = \text{constant}$ by the open mapping theorem

Write $f e^{-g} = e^c \Rightarrow f = e^{g-c}$ as claimed.

$\boxed{a} \Rightarrow \boxed{b}$ Let u harmonic. Let $F = u_x - i u_y$.

Claim F holomorphic

Indeed, F is of class C^1 & satisfies CR equations.

$$(u_x)_x = (-u_y)_y \iff u_{xx} + u_{yy} = 0 \quad \text{true}$$

$$(u_x)_y = -(-u_y)_x \iff u_{xy} = u_{yx}. \quad \text{true}$$

$\Rightarrow$ F holomorphic by Math 220, Lecture 2.

Since G is simply connected, F admits a primitive

$\Rightarrow$ $F = f'$ for f holomorphic, $f = \alpha + i\beta$.

$$f' = \alpha_x + i\beta_x = F = u_x - i u_y$$

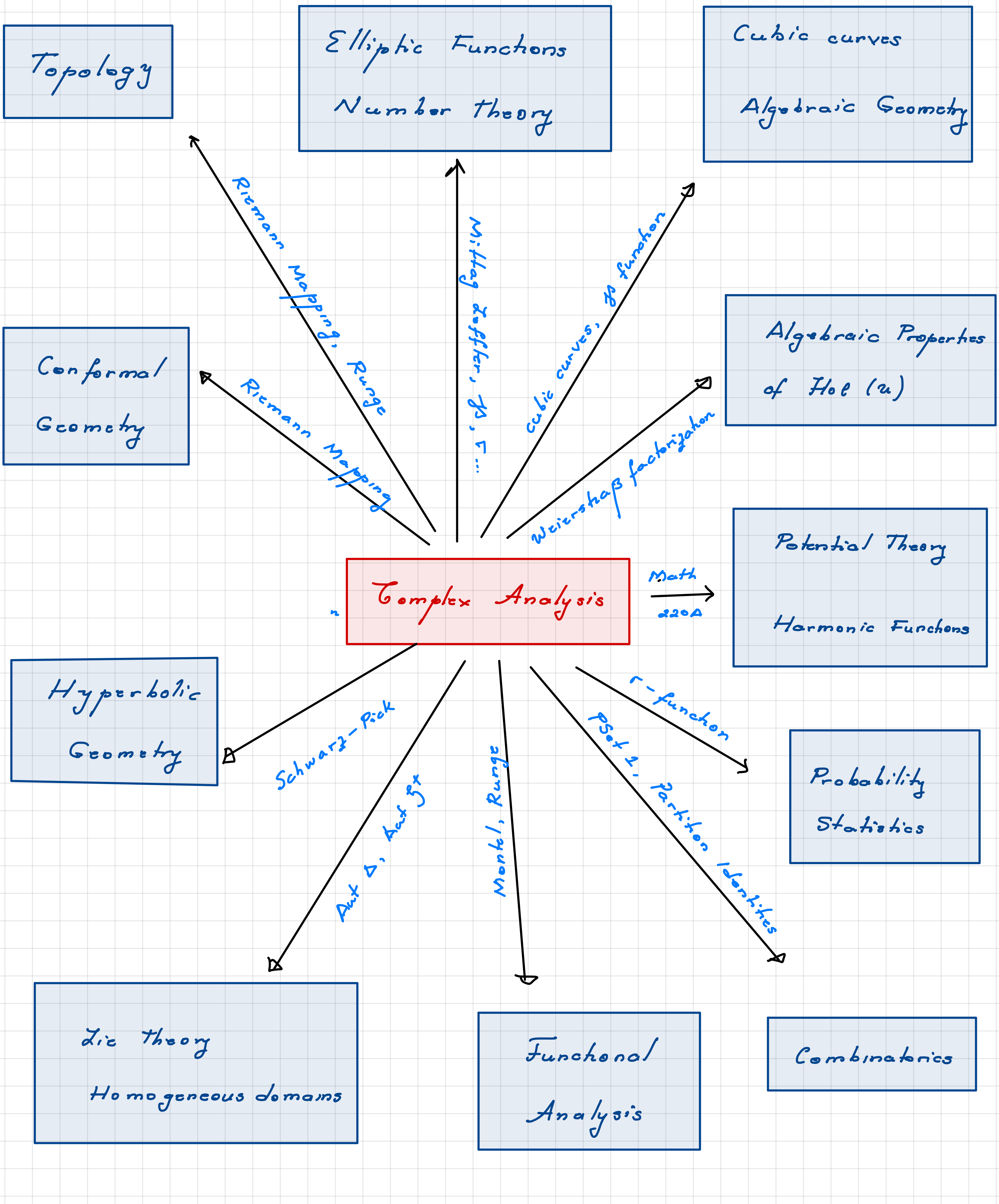
$$\Rightarrow \alpha_x = u_x \qquad \Rightarrow \alpha = u + C.$$

$$\Rightarrow \beta_x = -u_y = -\alpha_y \Rightarrow \alpha_y = u_y.$$

Cauchy-Riemann

Replacing f by $f - c$, we obtain $u = \operatorname{Re} f$ & $\beta = \operatorname{Im} f$ is the conjugate of u.

Summary of Math 220A – B



Math 220B Final Exam Review

To review, we list below the *Main Topics* covered in this class (this is not a comprehensive list):

(1) Products of holomorphic functions. Convergence, zeroes, logarithmic derivatives.
(2) Weierstraß elementary factors. Weierstraß factorization. Weierstraß problem.
(3) Mittag-Leffler problem in $\mathbb{C}$. Examples.
(4) Factorization of the sine function. The Gamma function.
(5) Normal families. Montel's theorem.
(6) Schwarz's lemma. Automorphisms of the disc. Schwarz-Pick.
(7) Riemann Mapping Theorem.
(8) Schwarz Reflection Principle.
(9) Runge's Theorem. Polynomial and rational approximation. Simple connectivity.

<u>Final Exam</u>

Monday, March 18, 11:30 – 2:30

<u>Office Hour</u>

Friday, March 15, 1 – 2:30

Final from 2021 & Solutions online

Additional Practice Problems