Math 220B - Lecture 2

January 10, 2024

1. Infinite Products of Holomorphic Functions

Recall $\sum_k |a_k|$ converges $\iff \prod_k (1+a_k)$ converges absolutely. $(+)$

Let $f_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\mathcal{U} \subseteq \mathbb{C}$

Assumption $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

Terminology $\sum_{k=1}^{\infty} f_k$ converges absolutely locally uniformly.

Define $(*)$ $F(z) = \prod_{k=1}^{\infty} (1 + f_k(z)).$

Remark $(*)$ converges absolutely for each fixed $z \in \mathcal{U}$

$\rightsquigarrow$ we can rearrange.

Remark In practice, instead of Assumption we may check

$$\sum_{n=1}^{\infty} \sup_K |f_n| < \infty \quad \forall\, K \subseteq \mathcal{U} \text{ compact}$$

$\rightsquigarrow$ "normal convergence"

Conway VII 5. 9.

Proposition Assume $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly.

[i] the partial products of (*) converge locally uniformly. to F

[ii] F is holomorphic

[iii] $F(z_0) = 0 \iff \exists\, k$ with $1 + f_k(z_0) = 0$

Furthermore,

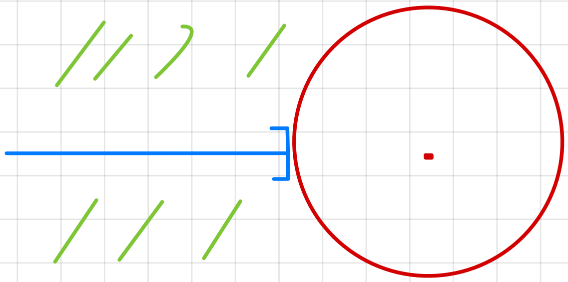
$$\operatorname{ord}(F, z_0) = \sum_{k=1}^{\infty} \operatorname{ord}(1 + f_k, z_0).$$

Example $z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$ defines an entire function

with zeroes only at the integers. & nowhere else.

Indeed, apply the Proposition to $f_k(z) = \frac{z^2}{k^2}$.

Proof Recall from last time



$\mathrm{Log}(1+z)$ is continuous in $\Delta(0,1)$

Important inequality: $\exists \rho < 1$ such that

$$|\mathrm{Log}(1+z)| \le \frac{3}{2}|z| \quad \text{if } |z| \le \rho$$

Proof of (ii) Let $K \subseteq \mathcal{U}$ compact. Note that

$\sum |f_n|$ converges uniformly on K $\Rightarrow$ $f_n \rightrightarrows 0$ on K

$\Rightarrow$ $\forall \rho$ $\exists N$ with $|f_n| < \rho$ $\forall n \ge N$, on K.

$\Rightarrow$ by important inequality

$$\left|\mathrm{Log}(1+f_k(z))\right| \le \frac{3}{2}|f_k(z)| \quad \text{for } z \in K,\ k \ge N.$$

Since $\sum_{k=N}^{\infty} |f_k|$ converges uniformly by assumption,

comparison to test $\Longrightarrow$ $\sum_{k=N} \log(1+f_k(z))$ converges (absolutely) uniformly on K

Write $G_n = \sum_{k=N}^{n} \log(1+f_k(z)) \underset{K}{\rightrightarrows} G$.

Note that G_n is continuous since $\log(1+w)$ is continuous for $|w| < \rho$.

Since $G_n \underset{K}{\rightrightarrows} G$, by the claim $\boxed{a}$ below

$e^{G_n} \underset{K}{\rightrightarrows} e^{G} \Longrightarrow \prod_{k=N}^{n} (1+f_k(z)) \underset{K}{\rightrightarrows} e^{G}$.

$\Longrightarrow \prod_{k=1}^{n} (1+f_k(z)) \underset{K}{\rightrightarrows} e^{G} (1+f_1(z)) \cdots (1+f_{N-1}(z))$. $(+)$

(Uniform convergence after multiplication uses claim $\boxed{b}$ below)

Thus

$$F = e^{c}(1+f_1)\dots(1+f_{N-1}) \text{ in } K. \ \& \text{ the convergence is}$$

uniform, in K, completing the proof.

$\boxed{ii}$ F holomorphic by $\boxed{i}$ (local uniform convergence)

& Weierstraß Convergence theorem.

$\boxed{iii}$ Recall from last time that

$$F(z_0) = 0 \iff \exists k \text{ with } 1 + f_k(z_0) = 0.$$

To prove the assertion about orders, consider $(+)$

in $K = \bar{\Delta}$, Δ neighb. of z_0

Then,
$$F(z_0) = \underbrace{e^{G(z_0)}}_{\neq 0} (1+f_1(z_0)) \cdots (1+f_{N-1}(z_0))$$

$$\Rightarrow \operatorname{ord}(F, z_0) = \sum_{k=1}^{N-1} \operatorname{ord}(1+f_k, z_0)$$

$$= \sum_{k=1}^{\infty} \operatorname{ord}(1+f_k, z_0).$$

using that $1+f_k \neq 0$ for $k \geq N$ (because $|f_k| < \rho < 1$ in K.)

The following standard claims were used in the proof:

Claim [a] Let u_n be continuous, $u_n \underset{K}{\rightrightarrows} u$. Then $e^{u_n} \underset{K}{\rightrightarrows} e^{u}$.

[b] If $u_n \underset{K}{\rightrightarrows} u$, $v_n \underset{K}{\rightrightarrows} v$ (u_n, v_n continuous). Then

$$u_n v_n \underset{K}{\rightrightarrows} uv.$$

Note u, v continuous (Math 140B) hence bounded on K

Proof [a] Suffices to show $\sup_K |e^{u_n} - e^{u}| \longrightarrow 0$.

Compute

$$\sup_K |e^{u_n} - e^{u}| = \sup_K |e^{u}| \cdot |e^{u_n - u} - 1|$$

$$\leq \sup_K |e^{u}| \cdot \sup_K |e^{u_n - u} - 1|$$

$$= M \cdot \sup_K |e^{u_n - u} - 1| < \varepsilon M \quad \text{for } n \geq N.$$

Why? By continuity, $\exists\, \delta > 0$: $|e^{w} - 1| < \varepsilon$ if $|w| < \delta$.

Since $u_n \underset{K}{\rightrightarrows} u \Rightarrow \exists\, N$ with $|u_n - u| < \delta$ on K

$$\Rightarrow |e^{u_n - u} - 1| < \varepsilon$$

Proof of $\boxed{b}$ We show $\sup_K |u_n v_n - uv| \longrightarrow 0$.

Indeed by triangle inequality

$$\sup_K |u_n v_n - uv| \leq \sup_K |(u_n - u)(v_n - v)| + \sup_K |u(v_n - v)| + \sup_K |v(u_n - u)|$$

$$\leq \sup_K |u_n - u| \cdot \sup_K |v_n - v| + \sup_K |u| \cdot \sup_K |v_n - v| + \sup_K |v| \cdot \sup_K |u_n - u|$$

$\longrightarrow 0$ since $\sup_K |u_n - u| \longrightarrow 0$ and $\sup_K |v_n - v| \longrightarrow 0$.

Logarithmic derivative

Taking derivatives of products is messy. It is easier to take logarithmic derivatives

h holomorphic $\Rightarrow$ $\frac{h'}{h}$ = logarithmic derivative

= holomorphic away from Zero(h)

Addition formula

$h = fg \Longrightarrow \frac{h'}{h} = \frac{f'}{f} + \frac{g'}{g}.$

$h' = f'g + fg' \Rightarrow \frac{h'}{h} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$

Inductively $\quad h = f_1 \cdots f_s \Rightarrow \frac{h'}{h} = \frac{f_1'}{f_1} + \cdots + \frac{f_s'}{f_s}$

We prove the same for infinite products. Let

$\boxed{\text{i}}$ $g_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f_k = 1 + g_k$. s.t.

$\boxed{\text{ii}}$ $\sum_{k=1}^{\infty} |g_k|$ converges locally uniformly in $\mathcal{U}$

Proposition $h = \prod_{k=1}^{\infty} f_k$. Away from $Zero(h)$:

$$\frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

The RHS converges locally uniformly on $\mathcal{U} \setminus Zero(h)$.

Proof Recall from (+) in the previous Proof that

for $K = \bar{\Delta} \subseteq \mathcal{U} \setminus Zero(h)$, Δ neighborhood of an arbitrary point in $\mathcal{U} \setminus Zero(h)$, $\exists\ N$ with

$$F_n = \prod_{k=N}^{n} f_k \xrightarrow[\Delta]{l.u.} F = e^G \quad \text{on } \Delta$$

$$G = \sum_{k=N}^{\infty} \log(1 + g_k)$$

Note $h = f_1 \cdots f_{N-1} \prod_{k=N}^{\infty} f_k = f_1 \cdots f_{N-1} \, e^G$

finite case

$$\Rightarrow \frac{h'}{h} = \frac{f_1'}{f_1} + \cdots + \frac{f_{N-1}'}{f_{N-1}} + \frac{(e^G)'}{e^G}.$$

$$\Rightarrow \frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

We need to show $\displaystyle\sum_{k=N}^{n} \frac{f_k'}{f_k} \xrightarrow[\triangle]{\ell.u.} \frac{(e^G)'}{e^G}.$

To see this, recall that by the important inequality & comparison test:

$$\sum |g_k| \text{ converges unif. on } K \Rightarrow \sum_k \mathrm{Log}(1+\underbrace{g_k}_{f_k}) \text{ converges}$$

(absolutely) uniformly.

Also $\mathrm{Log}\,(\underbrace{1+g_k}_{f_k})$ is holomorphic ($|g_k| < \rho$ on K)

By Weierstraß convergence

$$\sum_{k=N}^{\infty} \log f_k \overset{l.u.}{=} G \Rightarrow \sum_{k=N}^{\infty} (\mathrm{Log}\, f_k)' \overset{l.u.}{=} G'$$

$$\Rightarrow \sum_{k=N}^{\infty} \frac{f_k'}{f_k} \overset{\ell.u.}{=} G'$$

Factorization of sine (Euler, 1734 / Conway VII.6)

"De Summis Serierum Reciprocarum"

Theorem

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

Idea. Both sides have the same zeroes (with multiplicity)

HWK 1, #4

Lemma If $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ entire have the same zeros (w/ multiplicity) then $f = g e^h$ for some entire function h.

Proof of the sine factorization

(1) convergence:

Note that $\sum_{k=1}^{\infty}\left|\frac{z^2}{k^2}\right|$ converges locally uniformly $\Rightarrow \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ converges.

(2) location of zeroes:

Both sides $\sin \pi z$ & $\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ have simple zeroes at the integers & nowhere else.

(3) completing the proof

By the Lemma, $\exists$ h entire

$$\sin \pi z = e^{h}\, \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$$

We show $h \equiv 0$. Compute logarithmic derivative

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{(e^h)'}{e^h} + \frac{\pi}{\pi z} + \sum_{k=1}^{\infty} \frac{-\frac{2z}{k^2}}{1-\frac{z^2}{k^2}}$$

$$\pi \cot \pi z = h' + \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}$$

Recall Math 220, Hwk 6:

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2} \, dz$$

via the residue theorem. Making $n \to \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

Thus $h' \equiv 0 \Rightarrow h$ constant. We show $e^{h} \equiv 1$.

From

$$\frac{\sin \pi z}{\pi z} = e^{h} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \quad \text{make } z \to 0$$

$$\downarrow$$

$$1 = e^{h(0)} \cdot 1 \Rightarrow e^{h} = e^{h(0)} \equiv 1.$$

This completes the proof.

Remark

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

$\boxed{\text{I}}$ $z = \frac{1}{2}$

$$1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left(1 - \frac{1}{4k^2}\right) = \frac{\pi}{2} \cdot \prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{(2k)(2k)}$$

$$\Rightarrow \frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)(2k)}{(2k-1)(2k+1)}$$

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$ Wallis, 1655

$\boxed{\text{II}}$ $z = i$

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2}\right) = \frac{\sin \pi i}{\pi i} = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

$\boxed{\text{III}}$ $$\cos \pi z = \frac{\sin 2\pi z}{2 \sin \pi z} = \frac{2\pi z \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{k^2}\right)}{2\pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}$$

Splitting into k even / odd:

$$\cos \pi z = \prod_{l=1}^{\infty} \left(1 - \frac{4z^2}{(2l-1)^2}\right).$$