Math 220 B — Lecture 3

January 17, 2024

1. Γ- function

— probability, statistics, combinatorics, ...

↳ Conway VII. 7.

" The product $1.2.\ldots.x$ is the function that must be introduced in analysis " (Gauss to Bessel, 1811)

$$\Pi x = "1.2.3\ldots.x" = \Gamma(x+1)$$

" The theory of analytic factorials does not seem to have the importance some mathematicians used to attribute to it"

Weierstraß 1854

Definition $\quad G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$

Remark The convergence (absolutely & locally uniformly) of the product is HWK 1, # 2. There, you show

$$\sum_{n=1}^{\infty}\left|\log\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right]\right| \quad \text{converges locally uniformly.}$$

Properties of the function G

i $G(z)\,G(-z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{z/n}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{-z/n}$

$$= \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right)e^{z/n}e^{-z/n}$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z} \quad \text{(sine factorization)}$$

ii $G(z-1) = z\,G(z)\,e^{\gamma}$ where γ is constant.

iii $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right).$ ← see below

. .

Definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\cdot\frac{1}{G(z)}$$

Remark G has zeroes at $-1, -2, \dots -n, \dots$

$\Rightarrow \Gamma$ meromorphic in $\mathbb{C}$ with zeroes at $-1, -2, \dots, -n, \dots$

Proof of $\boxed{ii}$

We inspect zeroes of both sides.

Zeroes G : $\quad -1, -2, \dots, -n, \dots$

$$\left.\begin{array}{ll} zG(z): & 0, -1, -2, \dots, -n, \dots \\[2ex] G(z-1): & 0, -1, -2, \dots, -n \end{array}\right\} \text{ have the same zeroes}$$

$\Longrightarrow \quad G(z-1) = z\,G(z)\,e^{\gamma(z)}$ for some function $\gamma(z)$.

We need $\gamma(z) = \text{constant}$. We verify $\gamma' = 0$.

Take logarithmic derivatives

$$\frac{G'(z-1)}{G(z-1)} = \frac{1}{z} + \frac{G'(z)}{G(z)} + \gamma' \qquad (*)$$

Recall that

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Since logarithmic derivative turns products into sums

$$\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty} \left(\frac{\frac{1}{n}}{1 + \frac{z}{n}} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

$$\Longrightarrow \frac{G'(z-1)}{G(z-1)} = \sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n} \right) = \left(\frac{1}{z} - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n+1} \right)$$

$n \geq 2 \quad n \to n+1$

$$= \left(\frac{1}{z} - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)}$$

$$= \left(\frac{1}{z} - \cancel{1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \cancel{1}$$

$$= \frac{1}{z} + \frac{G'(z)}{G(z)} \quad (*) \quad \Rightarrow \gamma'(z) = 0 \quad \Rightarrow \gamma(z) = \gamma. = \text{constant}$$

$\boxed{iii}$ The above constant is

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log(n+1) \right) \quad \text{(Euler constant)}$$

Indeed, $G(0) = 1$ by definition of the function G.

By $\boxed{iii}$ $\Longrightarrow$ $G(z-1) = z\, G(z)\, e^{\gamma} \overset{z=1}{\Longrightarrow} G(1) = e^{-\gamma}$.

Using the definition

$$G(1) = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-1/k} =$$

$$= \lim_{n \to \infty} \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) e^{-1/k} =$$

$$= \lim_{n \to \infty} \frac{n+1}{\cancel{n}} \cdot \frac{\cancel{n}}{\cancel{n-1}} \cdot \cancel{\ } \cdots \cancel{\ } \cdot \frac{\cancel{2}}{1} \cdot e^{-1 - 1/2 - \dots - 1/n}$$

$$= \lim_{n \to \infty} e^{-\left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log(n+1)\right)} = e^{-\gamma}.$$

$$\Longrightarrow \gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log(n+1)\right)$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right)$$

Definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)}$$

Properties of Γ

$\boxed{i}$ $\quad \Gamma(1) = \frac{e^{-\gamma}}{G(1)} = 1$ using $G(1) = e^{-\gamma}$ from above.

$\boxed{ii}$ $\quad \Gamma(z+1) = z\,\Gamma(z) \rightsquigarrow$ "Γ behaves like a factorial"

In particular, by induction

$$\Gamma(n) = (n-1)! \qquad \forall\, n > 0,\ n \in \mathbb{Z}.$$

This follows by direct computation

$$\Gamma(z+1) = \frac{e^{-\gamma z - \gamma}}{(z+1)} \cdot \frac{1}{G(z+1)} \overset{???}{=} \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)} \cdot z = z\,\Gamma(z)$$

$\Longleftrightarrow \quad G(z) = (z+1)\,G(z+1)\,e^{\gamma}$ which is true.

(see above)

$\boxed{iii}$ $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$. In particular $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Compute $\Gamma(z)\,\Gamma(1-z) \overset{\boxed{ii}}{=} \Gamma(z)\,(-z)\,\Gamma(-z) =$

$$\xrightarrow{\text{definition}} = \frac{e^{-\gamma z}}{z\,G(z)} \cdot \cancel{(-z)} \cdot \frac{e^{\gamma z}}{\cancel{(-z)}\,G(-z)}$$

$$= \frac{1}{z\,G(z)\,G(-z)} = \frac{\pi}{\sin \pi z} \quad \text{(see above)}$$

$\boxed{IV}$ $$\Gamma(z) = \lim_{n\to\infty} \frac{n!}{z\,(z+1)\,\dots\,(z+n)}\, n^{z}$$ (Gauss' definition)

We use the definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\,\frac{1}{G(z)} =$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} e^{z/k}$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \cdot \prod_{k=1}^{n} \frac{k}{z+k} \cdot e^{z\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n!}{z\,(z+1)\,\dots\,(z+n)} \cdot e^{z\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n-\gamma\right)}_{\downarrow\ 0}} \cdot n^{z}$$

$$= \lim_{n\to\infty} \frac{n!}{z\,(z+1)\,\dots\,(z+n)} \cdot n^{z}.$$

Exercise (Conway VII. 7.3)

Legendre duplication formula.

Use Gauss' definition to check

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

Residues

Note that $\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)}$ is

meromorphic with poles at $0, -1, -2, \dots$ since G has

zeroes at $-1, -2, \dots$

What are the residues?

$$\mathrm{Res}(\Gamma, -n) = \lim_{z \to -n} (z+n)\, \Gamma(z) =$$

$$\boxed{\text{ii}}$$

$$= \lim_{z \to -n} \cancel{(z+n)} \frac{\Gamma(z+n+1)}{z(z+1)\dots\cancel{(z+n)}}$$

$$= \lim_{z \to -n} \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n-1)}$$

$$= \frac{\Gamma(1)}{(-n)\dots(-1)} = \frac{1}{(-1)^n n!} = \frac{(-1)^n}{n!}$$

Remark It can be shown

Mellin transform of e^{-t}

$$\boxed{\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt} \quad \text{for } \operatorname{Re} z > 0.$$

Step 1 Convergence of RHS. Fix z, $\operatorname{Re} z > 0$.

$$\left| \int_0^1 e^{-t}\, t^{z-1}\, dt \right| \le \int_0^1 |e^{-t}\, t^{z-1}|\, dt \qquad \text{since } |e^{-t}| \le 1$$

$$\le \int_0^1 t^{\operatorname{Re} z - 1}\, dt$$

$$= \frac{t^{\operatorname{Re} z}}{\operatorname{Re} z} \Big|_{t=0}^{t=1} = \frac{1}{\operatorname{Re} z} \quad \text{using } \operatorname{Re} z > 0.$$

Pick A with $|t^{z-1}| \le e^{t/2}$ when $|t| > A$

$$\left| \int_A^\infty e^{-t}\, t^{z-1}\, dt \right| \le \int_A^\infty |e^{-t}\, t^{z-1}|\, dt \le \int_A^\infty e^{-t}\, e^{t/2}\, dt$$

$$= \int_A^\infty e^{-t/2}\, dt$$

$$= 2e^{-A/2} < \infty.$$

$$\int_1^A e^{-t}\, t^{z-1}\, dt < \infty \quad \text{by continuity of } e^{-t}\, t^{z-1} \text{ in } t.$$

Step 2 Using integration by parts, one easily shows

$$\int_0^n \underbrace{\left(1-\frac{t}{n}\right)^n}_{u} \underbrace{t^{z-1}\,dt}_{dv} = \frac{n^z \quad n!}{z\,(z+1)\cdots(z+n)}$$

Exercise – check the details.

Step 3 Make $n \to \infty$. From real analysis

$\left(1-\frac{t}{n}\right)^n \longrightarrow e^{-t}$ as $n \to \infty$. This will also be explained below. We will argue that

$$(1) \quad \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt \xrightarrow{n\to\infty} \int_0^\infty e^{-t}\, t^{z-1}\,dt$$

$\parallel$ Step $\boxed{ii}$

$$\frac{n^z \quad n!}{z\,(z+1)\cdots(z+n)} \xrightarrow{n\to\infty} \Gamma(z) \text{ by Gauss' formula}$$

This shows $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\,dt.$

Rigorous justification of convergence in (1)

Claim $0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{e^{-t}\, t^2}{n}$ if $0 \le t \le n$.

Assuming the claim, we prove Step 3. Compute

$$\int_0^\infty e^{-t}\, t^{z-1}\, dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt =$$

$$= \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1} dt + \int_n^\infty e^{-t}\, t^{z-1} dt \longrightarrow 0$$

We claim the terms converge to 0.

term II. $\int_n^\infty e^{-t}\, t^{z-1} dt \longrightarrow 0$ as $n \to \infty$ because

$\int_0^\infty e^{-t}\, t^{z-1} dt$ converges by Step 1.

term II

$$\left| \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n \right) t^{z-1} dt \right| \overset{\text{claim}}{\leq}$$

$$\leq \int_0^n \frac{1}{n} t^2 e^{-t} \cdot t^{\operatorname{Re} z-1} dt < \int_0^\infty \frac{1}{n} \cdot t^2 e^{-t} t^{\operatorname{Re} z+1} dt$$

$$= \frac{1}{n} \underbrace{\int_0^\infty e^{-t} t^{\operatorname{Re} z+1} dt}_{\text{converges by step 1.}} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof of claim

(a) first inequality

Use $1-y \leq e^{-y}$ for $y \geq 0$.

Take

$$y = \frac{t}{n} \Rightarrow 1 - \frac{t}{n} \leq e^{-t/n} \Rightarrow \left(1-\frac{t}{n}\right)^n \leq e^{-t}.$$

To see $1-y \leq e^{-y}$, let $f(y) = e^{-y} - (1-y)$,

$f(0) = 0$, $f' = -e^{-y} + 1 > 0 \Rightarrow f \nearrow \Rightarrow f(y) \geq f(0) = 0$

(b) second inequality.

The inequality to prove is

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n} \Longleftrightarrow$$

$$\Longleftrightarrow \quad 1 - e^t \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{n} \qquad (*)$$

Use $e^y \geq 1 + y$ for $y \geq 0$ proven just as above. Take $y = \frac{t}{n}$

Since $e^t = \left(e^{t/n}\right)^n \geq \left(1 + \frac{t}{n}\right)^n$, to show $(*)$ we show

$$1 - \left(1 + \frac{t}{n}\right)^n \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{n} \Longleftrightarrow$$

$$\Longleftrightarrow \quad \left(1 - \frac{t^2}{n}\right) \leq \left(1 - \frac{t^2}{n^2}\right)^n. \quad \text{This is true.}$$

Indeed, use $(1-y)^n \geq 1 - ny$ for $y = \frac{t^2}{n^2}$

The last inequality can be proved by induction on n.