Math 220 B — Lecture 4

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The Weierstraß Problem

Conway VII.5

Given $\boxed{i}$ $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

$\boxed{ii}$ $\{m_n\}$ positive integers

Find entire functions f with zeroes only at a_n of order m_n.

Remark This also makes sense for arbitrary regions $\mathcal{U} \subseteq \mathbb{C}$

Main Theorem

The Weierstraß problem is always solvable in $\mathbb{C}$.

Corollary Every meromorphic function in $\mathbb{C}$ is quotient of two entire functions.

Proof Let h be meromorphic. Let P be the collection of poles of h listed with multiplicity. Let g be the solution to the Weierstrass problem for P. (The set P has no limit point in $\mathbb{C}$, so the hypothesis of Weierstrass is satisfied.) Then $f = gh$ entire. &

see Remark $\boxed{16}$ below

$$h = \frac{f}{g}$$

Remarks [i] Any two solutions f_1 & f_2

$$f_1 = e^{h} f_2 \, , \quad h \text{ entire.}$$

[ii] If $\{a_n\}$ has no limit point in $\mathbb{C}$ then $a_n \longrightarrow \infty$.

Indeed, if not, $\exists\, r > 0$ such that $\forall\, N \;\; \exists\, n \geq N, \;\; |a_n| \leq r$.

This means $\exists$ subsequence of $\{a_n\}$ bounded by r. Since $\overline{\Delta}(0,r)$ compact, this will have a convergent subsequence, with limit $a \in \mathbb{C}$. $\rightsquigarrow$ contradiction (Lecture 8, 220 A)

[iii] Repetitions & zero terms.

We will agree from now on that $\{a_n\}$ may contain repetitions. That is, by relabelling we can repeat each zero as many times as their multiplicity.

We assume $a_n \neq 0 \;\; \forall\, n$. If 0 is a zero for f, we will add it via multiplication by z^m at the end.

Solution to the Weierstraß Problem

Naive attempt $f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$

Issue: Convergence!

Idea: Try $f(z) = \prod_{n=1}^{\infty} f_n(z)$ where

f_n has zero at a_n. e.g. $f_n(z) = \left(1 - \frac{z}{a_n}\right) e^{h_n}$

Hope $f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{h_n}$ converges.

Could this work? For example,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \text{ does not converge}$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} = G(z) \text{ does converge.}$$

Weierstraß elementary / primary factors

Define

$$E_p(z) = \begin{cases} 1-z & \text{if } p=0 \\ (1-z)\exp\left(z+\frac{z^2}{2}+\dots+\frac{z^p}{p}\right) & \text{if } p>0. \end{cases}$$

$\implies E_p$ is entire with a simple zero at $z=1$.

Remark $E_p\left(\frac{z}{a}\right)$ has a simple zero at $z=a$.

We look for an answer of the form

zeros at a_n.

$$(*) \quad f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \quad \text{for suitable } p_n \geq 0$$

Issue: Can we pick p_n such that $(*)$ converges absolutely & locally uniformly.

Recall: $\sum_{n=1}^{\infty} |f_n|$ converges locally uniformly $\Rightarrow \prod_{n=1}^{\infty}(1+f_n)$ converges absolutely locally uniformly.

We wish to use this for $f_n = E_{p_n}\left(\frac{z}{a_n}\right) - 1$.

Growth of the elementary factors

Lemma $|1 - E_p(z)| \leq |z|^{p+1}$ if $|z| \leq 1$.

Proof later.

Lemma Given $a_n \longrightarrow \infty$, $a_n \neq 0$, $\exists\ p_n$ natural numbers (not unique) such that

$$\forall\, r > 0 \implies \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n + 1} < \infty.$$

Proof

For instance, take $p_n = n-1$. Let $r > 0$.

Since $a_n \longrightarrow \infty$, $\exists\ N$ such that $|a_n| \geq \frac{r}{2}$ if $n \geq N$.

$$\implies \frac{r}{|a_n|} \leq \frac{1}{2} \implies \left(\frac{r}{|a_n|} \right)^n \leq \frac{1}{2^n}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \implies \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^n < \infty$. by comparison test.

Theorem (Weierstraß)

Let $a_n \to \infty$, $a_n \neq 0$. Pick p_n as in the previous Lemma:

$$\forall r > 0 \implies \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty.$$

Then $\displaystyle\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ converges absolutely & locally uniformly to an entire function w/ zeros only at a_n.

Proof

Let $f_n = E_{p_n}\left(\frac{z}{a_n}\right) - 1$. Pick K compact, $K \subseteq \Delta(0,r)$ for some r. We will argue that $\displaystyle\prod_{n=1}^{\infty}(1+f_n)$ converges locally uniformly.

It suffices to show $\displaystyle\sum_{n=1}^{\infty} |f_n|$ converges uniformly on $\Delta(0,r)$.

Note for $\Delta(0,r)$:

$$|f_n(z)| = \left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| \overset{1^{st}\ \text{Lemma}}{\leq} \left|\frac{z}{a_n}\right|^{p_n+1} \leq \left(\frac{r}{|a_n|}\right)^{p_n+1}.$$

This requires $\left|\frac{z}{a_n}\right| \leq \frac{r}{|a_n|} \leq 1$ which is true for $n \geq N$ since $a_n \to \infty$.

Since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \implies$ Weierstraß M-test $\displaystyle\sum_{n=1}^{\infty}|f_n|$ converges uniformly in $\Delta(0,r)$, as needed

$\rightarrow$ $\prod_{n=1} E_{p_n}\left(\frac{z}{a_n}\right)$ converges absolutely & locally uniformly

The statement about zeroes follows from Lecture 2 & the fact that $E_{p_n}\left(\frac{z}{a_n}\right)$ vanishes only at $z = a_n$.

Corollary Any (not identically 0) entire function can be written as

$$f(z) = z^m e^{h} \prod_n E_{p_n}\left(\frac{z}{a_n}\right), \quad h = \text{entire}.$$

The choices of p_n & h are not unique.

↳ Weierstraß factorization

Remark For the same function f, several p_n's may work.

Changing p_n into $\tilde{p}_n$ can be absorbed in the exponential.

Proof WLOG we may assume $f(0) \neq 0$. Else if $\operatorname{ord}(f,0) = m$ we add the factor z^m.

Let $\{a_n\}$ be the zeroes of f listed with multiplicity.

Both f and $\prod_n E_{p_n}\left(\frac{z}{a_n}\right)$ solve the Weierstraß problem.

Apply Remark $\boxed{1}$ to conclude.

see HWK 2

Remark Weierstraß' theorem allows us to define functions which were not even thinked at before.

Poincaré: "Weierstraß' most important contribution to the theory of complex variables is the discovery of primary factors."

Analogy with number theory

The factorization.

$$f(z) = z^m e^h \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is reminiscent of the factorization of integers into primes.

$$\text{primes} \longleftrightarrow E_p$$

$$\text{units} \longleftrightarrow e^h$$

Difference. Not canonical / uniqueness of p_n's.

We can however ask questions with arithmetic flavor.

HWK 2, #2.

Example

$\boxed{i}$ $Q(z) = \prod_{k=1}^{\infty} (1+q^k z) = \prod_{k=1}^{\infty} E_0(-q^k z)$, $|q|<1$.

Note $p_k = 0$ & $\sum_{k=1}^{\infty} (r\, q^k)^{p_k+1} < \infty$. so the hypothesis of Weierstrass factorization holds.

$\boxed{ii}$ $G(z) = \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}} = \prod_{k=1}^{\infty} E_1\left(-\frac{z}{k}\right)$.

Note $p_k = 1$ & $\sum \left(\frac{r}{k}\right)^{p_k+1} < \infty$.

$\boxed{iii}$ $\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$

$$= \pi z \prod_{k=1}^{\infty} \left(1-\frac{z}{k}\right) e^{z/k} \cdot \left(1+\frac{z}{k}\right) e^{-z/k}$$

$$= \pi z \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} E_1\left(\frac{z}{k}\right)$$

$\boxed{iv}$ Do we get any new examples we didn't know?

Yes. See hwk for the Weierstrass σ- function.