Math 220 B — Lecture 5

January 24, 2024

Last time

I Define Weierstraß factors:

$$E_p(z) = \begin{cases} 1 - z & \text{if } p = 0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) & \text{if } p > 0. \end{cases}$$

II We saw that given $a_n \longrightarrow \infty$, $a_n \neq 0$,

$$f(z) = z^m e^h \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

are entire with zeroes at a_n.

III The p_n's are chosen so that

$$\forall\, r > 0, \quad \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n + 1} < \infty$$

Remark We have freedom in the choice of p_n.

Question Is there a canonical choice?

Assume $\exists\, h \in \mathbb{Z}_{\geq 0}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < \infty$.

If such h exists, pick the smallest one. This is called genus of the canonical product $\displaystyle\prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$

Example

[I] $$Q(z) = \prod_{k=1}^{\infty} (1+q^k z) = \prod_{k=1}^{\infty} E_0(-q^k z)$$

genus 0

[II] $$G(z) = \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k} = \prod_{k=1}^{\infty} E_1\left(-\frac{z}{k}\right)$$

genus 1

[III] $$\sigma = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right)$$ genus 2. (HWK 2, #4)

We still need to show:

Key Estimate

$$|1 - E_p(z)| \leq |z|^{p+1} \quad \text{for} \quad |z| \leq 1.$$

where $E_p(z) = (1-z)\, e^{u}$, $\quad u = z + \frac{z^2}{2} + \dots + \frac{z^p}{p}$

Proof Write $E_p(z) = \sum_{k=0}^{\infty} a_k z^k$.

By definition $E_p(0) = 1 \implies a_0 = 1$.

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$

Claim [i] $a_1 = a_2 = \dots = a_p = 0$

[ii] a_k real & $a_k \leq 0 \quad \forall\, k \geq p+1.$

[iii] $\sum_{k=p+1}^{\infty} a_k = -1.$

Assuming the Claim, we compute

$$|E_p(z) - 1| = \left|\sum_{k=1}^{\infty} a_k z^k\right| \overset{[i]}{=} \left|\sum_{k=p+1}^{\infty} a_k z^k\right|$$

$$= |z|^{p+1} \left|\sum_{k=p+1}^{\infty} a_k z^{k-p-1}\right|$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \, |z|^{k-p-1} \qquad |z| \leq 1$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k|$$

$$\overset{\boxed{ii}}{=} -|z|^{p+1} \sum_{k=p+1}^{\infty} a_k \overset{\boxed{iii}}{=} |z|^{p+1}$$

Proof of the claim

$\boxed{i}$ $\quad E_p(z) = (1-z)\, e^{u}, \qquad u = z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}.$

Note $u' = 1 + z + \dots + z^{p-1} \Rightarrow (1-z)\, u' = 1 - z^p.$

Compute

$$E_p'(z) = \left((1-z)\, e^{u}\right)' =$$

$$= -e^{u} + (1-z)\, u'\, e^{u}$$

$$= -e^{u} + (1-z^p)\, e^{u}$$

$$= -z^p e^{u} \qquad (1)$$

Since

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \Rightarrow E_p'(z) = \sum_{k=1}^{\infty} k\, a_k z^{k-1}. \qquad (2)$$

The terms in (1) have powers of z^p.

Comparing with (2) we see $a_k = 0 \quad \forall\ 1 \leq k \leq p$.

$\boxed{ii}$ Also for $k \geq p+1$,

$$a_k = -\frac{1}{k} \cdot \text{Coefficient of } z^{k-p-1} \text{ in } e^{u}.$$

Since

$$e^{u} = e^{z} \cdot e^{z^2/2} \cdot \ldots \cdot e^{z^p/p} \quad \& \text{ using the}$$

expansion of the exponential, we see that

real number

$$\text{Coefficient of } z^{k-p-1} \text{ in } e^{u} \geq 0 \implies a_k \leq 0.$$

$\boxed{iii}$ Set $z=1$:

$$0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k \implies \sum_{k=p+1}^{\infty} a_k = -1.$$

2. The Mittag - Leffler Problem

Conway VIII. 3 simplified.

Weierstraß Problem

Given (i) $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

(ii) $\{m_n\}$ positive integers

find entire functions f with zeroes only at a_n of order m_n.

Remark The function $1/f$ is meromorphic & its poles are only at a_n & their order equals m_n.

The Mittag - Leffler Problem asks a sharper question.

The Mittag-Leffler (ML) Problem for $\mathbb{C}$

Given [i] $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

[ii] Laurent principal parts (singular parts)

$$g_n(z) = \frac{A_{nm_n}}{(z-a_n)^{m_n}} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \cdots + \frac{A_{n1}}{z-a_n}$$

Main Theorem We can always find meromorphic function f with poles only at a_n & Laurent principal parts g_n near a_n.

Remark If f_1, f_2 are two solutions $\Rightarrow$ $f_1 - f_2$ = entire since the singular parts at a_n cancel out. Thus

$f_1 = f_2 + h$, h entire.

Remark This makes sense for $U \subseteq \mathbb{C}$.

- student of Hermite
  & Weierstraß
- Nobel Prize committee
- founder of Acta Math.

Gösta Mittag-Leffler

1846 – 1927

SUR LA REPRÉSENTATION ANALYTIQUE

DES

FONCTIONS MONOGÈNES UNIFORMES

D'UNE VARIABLE INDÉPENDANTE

PAR

G. MITTAG-LEFFLER
à STOCKHOLM.

Les recherches dont je vais exposer ici l'ensemble, ont été publiées auparavant, quant à leurs traits les plus essentiels, dans le Bulletin (Öfversigt) des travaux de l'Académie royale des sciences de Suède, ainsi que dans les Comptes-rendus hebdomadaires de l'Académie des sciences à Paris. Leur but est de faire parvenir, dans un certain sens, la théorie des fonctions analytiques uniformes d'une variable, à ce degré d'achèvement auquel la théorie des fonctions rationnelles est arrivée depuis longtemps.

Soit x une grandeur variable complexe à variabilité illimitée, et x' un point donné fini(¹) dans le domaine de la variable x. Soit enfin R une quantité positive donnée. Je dis que l'ensemble des points x remplissant la condition $|x - x'| < R$, constitue le *voisinage* ou *l'entourage* ou les *environs du point* x'(²) *correspondant à* R. Chacun de ces points est dit *appartenir au voisinage* ou *à l'entourage* ou *aux environs* R, ou être

(¹) C'est-à-dire représentant une valeur donnée finie.

(²) Cf.: *Zur Functionenlehre, von* K. WEIERSTRASS. Monatsbericht der Königl. Akademie der Wissenschaften zu Berlin, August 1880, pag. 4.

Acta Math 4 (1884)

Remarks Connections:

$$\begin{tikzcd}
 & \text{Generalized Weierstraß} \arrow[d, Rightarrow, "\text{Clear}"] \\
\text{Mittag Leffler} \arrow[ur, Rightarrow, "\text{HWK 2, \#6}"] \arrow[r, Rightarrow, "f \rightsquigarrow 1/f"'] & \text{Weierstraß Existence Problem} \\
 & \text{Weierstraß Factorization} \arrow[u, Rightarrow, "\text{Previous Lectures}"']
\end{tikzcd}$$

HWK 2, #6.

6. (*Generalized Weierstraß problem. Monday, January 25.*) Let $\{a_n\}$ be distinct complex numbers with $a_n \to \infty$. Fix complex numbers $\{A_n\}$. Show that there exists an entire function f such that

$$f(a_n) = A_n.$$

Further Connections

In HWK 2, Problem 4(iv) we will see that we can derive Mittag-Leffler for simple poles from Weierstraß factorization.

Discussion of the proof

Given $\{a_n\}$, $a_n \to \infty$, g_n = Laurent principal parts

we try $f = \sum_{n=1}^{\infty} g_n$ as solution to Mittag-Leffler

Issue: As usual, this may not converge

New idea: Pick h_n entire functions & argue

$$f = \sum_{n=1}^{\infty} (g_n - h_n) \text{ converges (away from poles).}$$

Since h_n are entire, we are not changing the Laurent principal parts.

Compare this to Weierstraß

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \quad \text{vs.} \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{h_n}$$

may not converge — could converge.

Terminology

$$\sum_{n=1}^{\infty} (g_n - h_n) = \text{Mittag-Leffler series}$$

h_n = convergence enhancing corrections

The h_n's are not unique!

Remark WLOG $a_n \neq 0$. $\forall n$.

The contributions of the poles at 0 are added at the end:

$$\frac{A_m}{z^m} + \cdots + \frac{A_1}{z} + \text{Solution with } a_n \neq 0.$$

Proof The proof is part of the theorem. Conway VIII. 3.

Fix $\boxed{1}$ $r_n \longrightarrow \infty$, $r_n < |a_n|$

$\boxed{11}$ c_n, $\sum_{n=1}^{\infty} c_n < \infty$

e.g. $c_n = \frac{1}{2^n}$, $c_n = \frac{1}{n^2}$, ...

Consider $$g_n(z) = \frac{A_{nm_n}}{(z-a_n)^{m_n}} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \cdots + \frac{A_{n1}}{z-a_n}$$

Since $a_n \neq 0$, g_n is holomorphic at $z=0$ in $\Delta(0, |a_n|)$

We can Taylor expand g_n in $\Delta(0, |a_n|)$ around 0.

Since $\bar{\Delta}(0, r_n) \subseteq \Delta(0, |a_n|)$, the Taylor series of g_n converges uniformly in $\bar{\Delta}(0, r_n)$. We can pick a Taylor polynomial h_n such that

$$|g_n - h_n| < c_n \quad \text{in } \bar{\Delta}(0, r_n).$$

Let $f = \sum_{k=1}^{\infty} (g_k - h_k)$. We show

Claim f meromorphic with poles only at a_k & principal parts g_k near a_k. $\Rightarrow$ f solves Mittag-Leffler.

Proof Let $r > 0$.

Since $r_k \longrightarrow \infty$, $\Rightarrow r_k > r$ if $k \geq N$. Then

$|g_k - h_k| < c_k$ in $\overline{\Delta}(0,r) \subseteq \Delta(0,r_k)$ if $k \geq N$..

By Weierstraß M-test $\sum_{k=N}^{\infty} (g_k - h_k)$ converges uniformly in $\overline{\Delta}(0,r)$. Note that since $|a_k| > r_k > r$

$\Longrightarrow$ $\underbrace{g_k}_{\text{the pole } a_k \text{ is not in } \Delta(0,r)} - \underbrace{h_k}_{\text{polynomial}}$ holomorphic in $\Delta(0,r)$. Thus the sum

$$\sum_{k=N}^{\infty} (g_k - h_k)$$

is holomorphic in $\Delta(0,r)$.

The sum $\sum_{k=1}^{N-1} (g_k - h_k)$ is meromorphic as a finite sum of meromorphic functions in $\Delta(0,r)$. The poles are only at those a_j's with $|a_j| < r$ and the Laurent principal parts are g_j. This is because h_k are polynomials. so they do not contribute to the Laurent principal parts.

$$\text{Thus } f = \overbrace{\sum_{k=1}^{N-1} (g_k - h_k)}^{\text{meromorphic}} + \overbrace{\sum_{k=N}^{\infty} (g_k - h_k)}^{\text{holomorphic}}$$

is meromorphic with poles at $|a_j| < r$ for all $\Delta(0,r)$.

Varying r we get the claim & finish the proof.

Summary of the proof

Step 1 Expand g_n into Taylor series at 0

Step 2 Pick h_n a Taylor polynomial & check

$$|g_n - h_n| < c_n \quad \text{in } \Delta(0, r_n) \quad \text{with } \sum_n c_n < \infty.$$

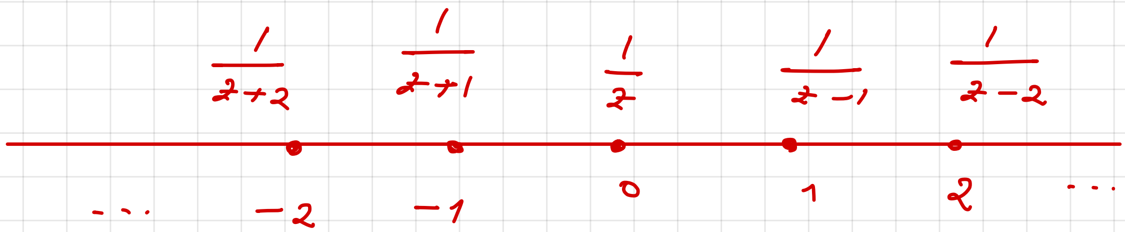
and $r_n < |a_n|$, $r_n \longrightarrow \infty$

Step 3 Solution

$$f = \sum_{n=1}^{\infty} (g_n - h_n) + \text{add Laurent principal part at } 0.$$

poles principal parts

Example [1] $\left(-n, \frac{1}{z+n}\right)$, $n \in \mathbb{Z}$.



Step 1 Taylor expand:

$$g_n = \frac{1}{z+n} = \frac{1}{n}\cdot\frac{1}{1+\frac{z}{n}} = \frac{1}{n}\left(1 - \frac{z}{n} + \frac{z^2}{n^2} - \cdots\right)$$

$$= \frac{1}{n} - \frac{z}{n^2} + \frac{z^2}{n^3} - \cdots$$

$$h_n = \frac{1}{n}. \quad , \; n \neq 0$$

Step 2 Let $r_n = \frac{1}{2}|n|^{1/2}$. If $|z| \le r_n$:

$$|g_n - h_n| = \left|\frac{1}{z+n} - \frac{1}{n}\right| = \frac{|z|}{|n|\,|n+z|} \le \frac{r_n}{|n|\,(|n| - r_n)} = c_n$$

Since $\lim\limits_{n\to\infty} \frac{c_n}{|n|^{-3/2}} < \infty$ and $\sum\limits_n \frac{1}{|n|^{3/2}} < \infty$. $\Longrightarrow \sum\limits_{n=1}^{\infty} c_n < \infty$.

Step 3   Mittag - Leffler solution

$$f = \sum_{n \neq 0} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \frac{1}{z}.$$

Collecting the terms for n & $-n$ we find

$$f = \sum_{n>0} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) + \frac{1}{z}$$

Math 220A, HWK6.

$$= \sum_{n>0} \frac{2z}{z^2 - n^2} + \frac{1}{z} = \pi \cot \pi z$$