Math 220 B — Lecture 6

January 29, 2024

Last time – Mittag-Leffler Problem in $\mathbb{C}$

Given

- $a_n \to \infty$ distinct and
- Laurent principal parts g_n

find f meromorphic with poles at a_n & principal parts g_n at a_n

Construction

Step 1 Expand g_n into Taylor series at 0

Step 2 Pick h_n a Taylor polynomial & check

$$|g_n - h_n| < c_n \quad \text{in } \Delta(0, r_n) \quad \text{with } \sum_n c_n < \infty.$$

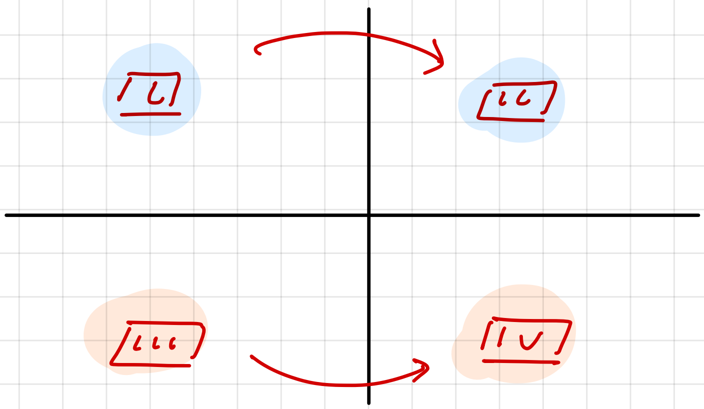
and $r_n < |a_n|$, $r_n \to \infty$

Step 3 Solution

$$f = \sum_{n=1}^{\infty} (g_n - h_n) + \text{add Laurent principal part at } 0.$$

Today – 4 historically important examples

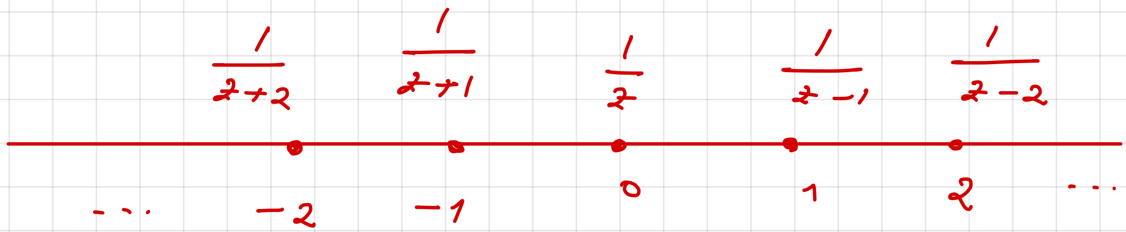
– we group them in pairs of two



We covered this last time already

poles principal parts

Example $\boxed{1}$ $\left(-n, \frac{1}{z+n}\right)$, $n \in \mathbb{Z}$.



Step 1 Taylor expand:

$$g_n = \frac{1}{z+n} = \frac{1}{n} \cdot \frac{1}{1+\frac{z}{n}} = \frac{1}{n}\left(1 - \frac{z}{n} + \frac{z^2}{n^2} - \dots\right)$$

$$= \frac{1}{n} - \frac{z}{n^2} + \frac{z^2}{n^3} - \dots$$

$$h_n = \frac{1}{n} \, , \quad n \neq 0$$

Step 2 Let $r_n = \frac{1}{2}|n|^{1/2}$. If $|z| \le r_n$:

$$|g_n - h_n| = \left| \frac{1}{z+n} - \frac{1}{n} \right| = \frac{|z|}{|n|\,|n+z|} \le \frac{r_n}{|n|\,(|n| - r_n)} = c_n$$

Since $\lim\limits_{n\to\infty} \frac{c_n}{|n|^{-3/2}} < \infty$ and $\sum\limits_n \frac{1}{|n|^{3/2}} < \infty$. $\implies \sum\limits_{n=1}^{\infty} c_n < \infty$.

Step 3 Mittag - Loffler solution

$$f = \sum_{n \neq 0} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \frac{1}{z}.$$

Collecting the terms for n & $-n$ we find

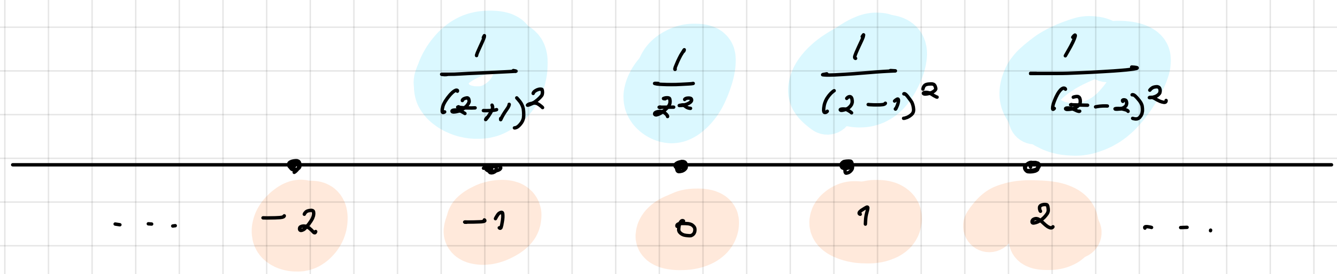
$$f = \sum_{n>0} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) + \frac{1}{z}$$

Math 220A, HWK6.

$$= \sum_{n>0} \frac{2z}{z^2 - n^2} + \frac{1}{z} = \pi \cot \pi z$$

$\boxed{11}$ Poles at $-n \in \mathbb{Z}$, principal parts $\dfrac{1}{(z+n)^2}$.

$$\left(-n, \frac{1}{(z+n)^2}\right)$$



Step 1

$$g_n = \frac{1}{(z+n)^2}$$

$$h_n = 0$$

Step 2 $\quad r_n = \frac{1}{2} |n|^{1/2}$. $\quad \forall\ |z| \le r_n$

$$|g_n - h_n| = \left| \frac{1}{(z+n)^2} \right| \le \frac{1}{(|n| - r_n)^2} = c_n .$$

$$\lim_{n\to\infty} \frac{c_n}{|n|^{-2}} = 1 \quad \& \quad \sum_{n\neq 0} \frac{1}{n^2} < \infty \implies \sum_{n \neq 0} c_n < \infty$$

Step 3 Mittag - Leffler function

$$f = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

We have seen $f = \dfrac{\pi^2}{\sin^2 \pi z}$ in Math 220A, HWK6, #7.

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2}\, dz$$

via the residue theorem. Making $n \to \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

7. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners

$$\pm\left(n + \frac{1}{2}\right) \pm ni.$$

Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{(z+a)^2}\, dz$$

via the residue theorem, and use this to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

HWK 6, Math 220A

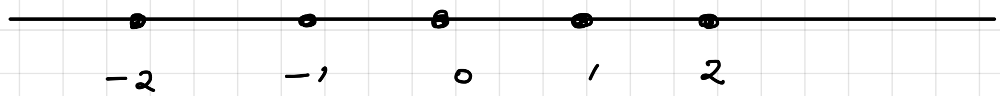
Remark Compare $\boxed{i}$ & $\boxed{ii}$

$$\left(-n,\ \frac{1}{z+n}\right) \longleftrightarrow \left(-n,\ \frac{1}{(z+n)^2}\right)$$

$$\pi \cot \pi z \longleftrightarrow \frac{\pi^2}{\sin^2 \pi z}$$

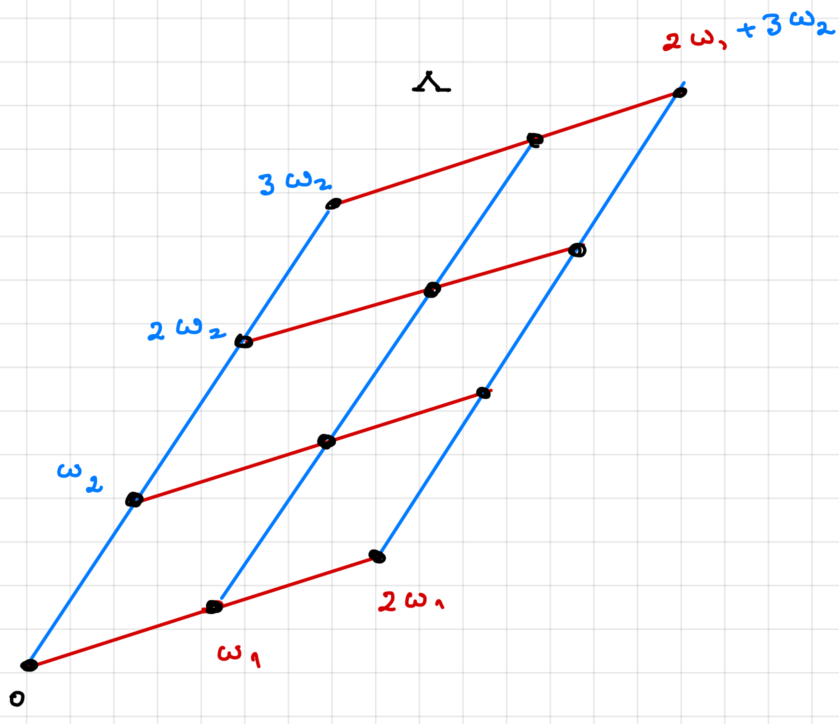
These are related by differentiation (up to a sign).

For the next examples, we replace

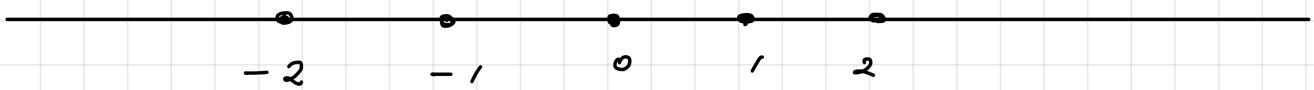


by the lattice

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{ m\,\omega_1 + n\,\omega_2 \; : m,n \in \mathbb{Z} \}, \quad \frac{\omega_1}{\omega_2} \notin \mathbb{R}$$



Main Difference



$$\sum_{n \neq 0} \frac{1}{|n|^{\alpha}} \quad \text{converges if } \alpha = 2$$

$$\text{if } \alpha > 1.$$

For the lattice,

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{|\lambda|^{\alpha}} \quad \text{converges if } \alpha = 3 \quad (\text{HWK } 2)$$

$$\text{if } \alpha > 2.$$

$\boxed{III}$

Poles at $\lambda \in \Lambda$, principal parts $\dfrac{1}{z-\lambda}$.

$$\left(\lambda, \frac{1}{z-\lambda}\right)_{\lambda \in \Lambda}.$$

Step 1 $\quad \lambda \neq 0$

Taylor expand

$$g_\lambda = \frac{1}{z-\lambda} = \frac{1}{\lambda} \cdot \frac{-1}{1-\frac{z}{\lambda}}$$

$$= \frac{-1}{\lambda}\left(1 + \frac{z}{\lambda} + \frac{z^2}{\lambda^2} + \dots\right)$$

$$= -\frac{1}{\lambda} - \frac{z}{\lambda^2} - \frac{z^2}{\lambda^3} - \dots$$

$$h_\lambda = -\frac{1}{\lambda} - \frac{z}{\lambda^2}$$

Step 2 $\quad$ Let $r_\lambda = \min\left(\frac{1}{2}|\lambda|, |\lambda|^{1/4}\right)$.

If $|z| \le r_\lambda$ then

$$|g_\lambda - h_\lambda| = \left|\sum_{k=2}^{\infty} \frac{z^k}{\lambda^{k+1}}\right|$$

$$= \frac{|z|^2}{|\lambda|^3} \sum_{k=0}^{\infty} \left|\frac{z}{\lambda}\right|^k \le \frac{r_\lambda^2}{|\lambda|^3} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} =$$

$$= 2 \cdot \frac{r_\lambda^2}{|\lambda|^3} \leq 2 \cdot \frac{1}{|\lambda|^{5/2}} = c_\lambda .$$

Since $\sum_{\lambda \neq 0} \frac{1}{|\lambda|^{5/2}} < \infty$. we get $\sum_{\lambda \neq 0} c_\lambda < \infty$.

Step 3 Mittag-Leffler solution

$$\zeta = \frac{1}{z} + \sum_{\lambda \neq 0} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right)$$

Weierstraß ζ- function (HWK 2, #4)

$\boxed{IV}$ Poles at $\lambda \in \Lambda$, principal parts $\dfrac{1}{(z-\lambda)^2}$.

$$\left(\lambda, \frac{1}{(z-\lambda)^2}\right)_{\lambda \in \Lambda}$$

Step 1 $\quad \lambda \neq 0$

$$g_\lambda = \frac{1}{(z-\lambda)^2} = \frac{1}{\lambda^2} \cdot \frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} =$$

$$= \frac{1}{\lambda^2}\left(1 + \frac{2z}{\lambda} + \frac{3z^2}{\lambda^2} + \cdots\right)$$

$$= \frac{1}{\lambda^2} + \frac{2z}{\lambda^3} + \frac{3z^2}{\lambda^4} + \cdots$$

$$\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \cdots$$

$$h_\lambda = \frac{1}{\lambda^2}.$$

Step 2 $\quad r_\lambda = \min\left(\frac{|\lambda|}{2}, |\lambda|^{1/4}\right)$

$$|h_\lambda - g_\lambda| = \left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{z^2 - 2z\lambda}{\lambda^2 (z-\lambda)^2}\right|$$

$r_\lambda \leq \frac{|\lambda|}{2}$

$$\leq \frac{r_\lambda^2 + 2 r_\lambda |\lambda|}{|\lambda|^2 (|\lambda| - r_\lambda)^2} \leq 4 \cdot \frac{r_\lambda^2 + 2 r_\lambda |\lambda|}{|\lambda|^4} = c_\lambda.$$

Note

$$\lim_{\lambda\to\infty} \frac{c_\lambda}{|\lambda|^{-5/2}} < \infty \implies \sum c_\lambda \sim \sum \frac{1}{|\lambda|^{5/2}} < \infty .$$

Step 3 The Mittag-Leffler solution

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) = \text{Weierstraß } \wp \text{ function.}$$

Homework 2, #5

Note that $\wp = -\zeta'$ as it should (the Laurent tails are related by differentiation, up to sign)

2. Further remarks*

↙ not needed for Qual

[i] A **divisor** is a formal sum

$$D = \sum_{p \in \mathbb{C}} n_p [p] \quad \text{where } n_p \in \mathbb{Z}$$

We require that this sum be locally finite. i.e.

$$\{p : n_p \neq 0\} \text{ does not accumulate}$$

Example $D = 3[p] + 5[q]$ is a divisor. $(p, q \in \mathbb{C})$

A divisor is non-negative (effective) if $n_p \geq 0 \ \forall p$.

Remark Divisors form a group under formal addition.

(just add the coefficients) $\rightsquigarrow$ group $\mathcal{D}$

e.g.

$$D_1 = [p] + 2[q]$$

$$D_2 = 3[q] \qquad \Rightarrow D_1 + D_2 = [p] + 5[q]$$

[ii] Any entire function gives rise to a divisor

Indeed,

$$\operatorname{div}(f) = \sum_{p \text{ zero for } f} \operatorname{ord}(f, p)\,[p]$$

Example

$$f = (z-a)^3 (z-b)^5 \Rightarrow \operatorname{div}(f) = 3\,[a] + 5\,[b]$$

[iii] Weierstraß Problem can be rephrased

Every effective divisor is the divisor on an entire function

$$D \geq 0\,, \quad D = \operatorname{div}(f).$$

[iv] For a meromorphic function f

$$\mathrm{div}(f) = \sum_{\substack{p \text{ zero or} \\ \text{pole}}} \mathrm{ord}(f,p)\,[p]$$

"principal divisor"

<u>Remark</u> $\mathrm{div}(fg) = \mathrm{div}\, f + \mathrm{div}\, g$

Principal divisors form a subgroup under addition.

↪ group $\mathcal{P}$

<u>Question</u> Is every divisor the divisor of a meromorphic function?

<u>Yes</u> For a general divisor D we can separate

$$D = D_+ - D_- \ , \quad D_+, D_- \text{ nonnegative.}$$

Write $D_+ = \mathrm{div}\, f_+$, $D_- = \mathrm{div}\, f_-$ & set $f = f_+/f_-$

Then $\quad \operatorname{div}(f) = \operatorname{div}(f_+) - \operatorname{div}(f_-)$ (check)

$$= D_+ - D_- = D.$$

Define

Divisor class group = Divisors / Principal Divisors $= \mathcal{D}/\mathcal{P}$

⇝ algebraic geometry

Weierstraß can be rephrased as

Divisor Class Group of $\mathbb{C}$ is trivial.

IV These questions naturally lead to sheaf cohomology.

"$H^1(\mathbb{C}, \mathcal{O}^*) = 0$"

Remark Both the Weierstraß problem & Mittag-Leffler problem can be solved for arbitrary regions $U \subseteq \mathbb{C}$, and for sequences of zeros/poles $\{a_n\}$ without accumulation points in U.

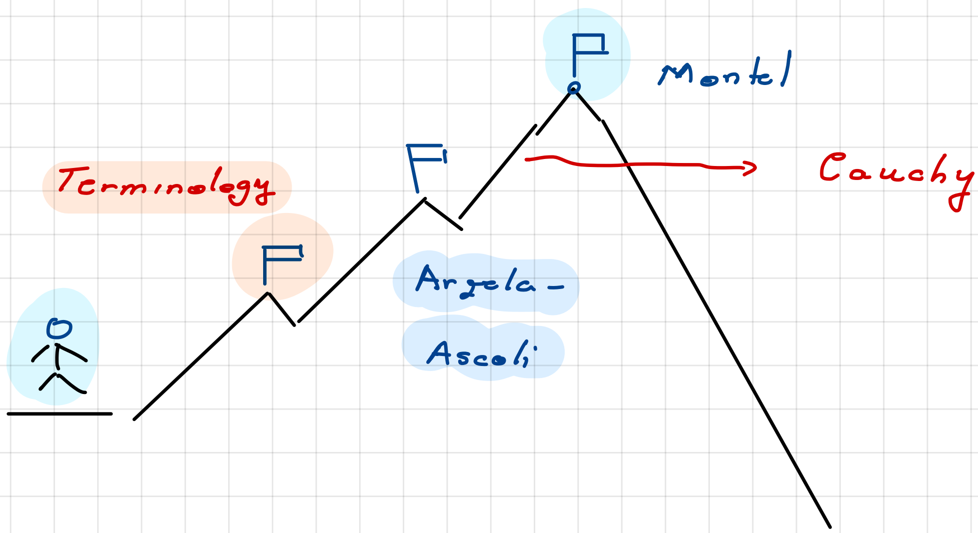
The arguments are more involved. (Conway VII.5.15, VIII.3.2)

Weierstraß Mittag-Leffler

As a result, the divisor class group of $U \subseteq \mathbb{C}$ is also trivial. In algebraic geometry, one encounters plenty of examples of nontrivial divisor class groups.

3. Next few lectures – Normal Families

Conway VII. 1 & 2.



Why climb the mountain – Motivation

Sequences of complex numbers

$\{a_n\}$ bounded $\Longrightarrow$ $\exists$ convergent subsequence

Indeed, if $|a_n| \leq M \Rightarrow a_n \in \bar{\Delta}(0, M)$. The closed disc $\bar{\Delta}(0, M)$ is compact.

We wish to make similar statements for sequences of functions (continuous or holomorphic).

Dream Statement

Given a "bounded" sequence of functions, there exists a "convergent" subsequence.

Question A

What could "—" mean?

Question B

Is this connected to compactness?

Answer is "yes" but it has no consequences for the current lecture.

Remark Dream statement makes sense in

[i] real analysis (continuous functions)

⇝ Arzela - Ascoli

[ii] complex analysis (holomorphic functions).

⇝ Montel.

We will investigate both.

Question A $f_n : U \longrightarrow \mathbb{C}$ "convergent" could mean

[i] pointwise ↙ weak

[ii] uniform ↙ strong

[iii] local uniform ↙ OK for us

$\Updownarrow$

[iv] uniform convergence on compact sets ↙ OK for us

"bounded" could mean

[i] pointwise bounded ← weak

$\forall x \in \mathcal{U} \;\; \exists M(x)$ with $|f_n(x)| < M(x) \;\; \forall n$

[ii] uniformly bounded ← strong

$\exists M \;\; \forall x \in \mathcal{U} \;\; |f_n(x)| < M \;\; \forall n$

[iii] locally uniformly bounded. ← OK for us

$\forall x \;\; \exists \Delta_x \subseteq \mathcal{U}$ neighborhood of x, such that the restrictions $f_n|_{\Delta_x}$ are uniformly bounded.

OK for us

[iv] uniformly bounded on compact sets

$\forall K \;\; \exists M(K), \quad |f_n(x)| \leq M(K) \;\; \forall x \in K \;\; \forall n$

Remark We have $\boxed{III} \Longleftrightarrow \boxed{IV}$ that is,

locally uniformly bounded $\Longleftrightarrow$

uniformly bounded on each compact

Why? $\Leftarrow$ If $x \in \mathcal{U}$, let $K = \overline{\Delta}_x$ be a compact neighborhood of x.

$\Rightarrow$ For all $x \in \mathcal{U}$, $\exists\ \Delta_x$ where $f_n\big/_{\Delta_x}$ are bounded by M_x.

Then $K \subseteq \bigcup\limits_{x \in K} \Delta_x \implies K \subseteq \bigcup\limits_{i} \Delta_{x_i}$ and let

$$M = \max(M_{x_1}, \dots, M_{x_n}) > 0.$$

This is a bound for all f_n's over K.

Dream Statement Revisited

$f_n : U \longrightarrow \mathbb{C}$ locally uniformly bounded

$\Longrightarrow$ f_n admits a locally convergent subsequence

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Question Could this be true?

Example No.

Let $U = \mathbb{R}$. The sequence

$$f_n(x) = \sin nx$$

is uniformly bounded, but we can't get a convergent subsequence

not even pointwise.

Question C1 Could this be true in complex analysis i.e. holomorphic functions? **YES**

Question C2 What is the correct statement in real analysis i.e. continuous functions?

Answer to C1

Main Theorem (Montel)

$f_n : U \longrightarrow \mathbb{C}$ holomorphic & locally uniformly bounde

$\Rightarrow$ f_n admits a locally uniformly convergent subsequence.