Math 220 B — Lecture 7

January 31, 2024

Dream Statement We wish to show

Main Theorem$^{-\varepsilon}$ (Montel)$^{-\varepsilon}$

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic & locally uniformly bounded

$\Rightarrow$ f_n admits a locally uniformly convergent subsequence.

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1. More generally – Families

$\mathcal{F}$ family of continuous or holomorphic functions.

Required for applications (Riemann-mapping & Picard's theorems)

Examples

[i] Any sequence determines $\mathcal{F} = \{f_1, f_2 \dots f_n \dots\} =$ family.

[ii] $\tilde{\mathcal{F}} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \ |a_k| \leq k \}$$

[iii] $\mathcal{F} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic, $f(0) = 1$, $\operatorname{Re} f > 0 \}$

Def $\mathcal{F}$ is normal if all sequences in $\mathcal{F}$ admit a locally uniformly convergent subsequence.

Remark The limit does not have to be in $\mathcal{F}$.

Example

$\boxed{1}$ $\mathcal{F}$ normal family of holomorphic functions

$\implies$ $\mathcal{F}'$ is normal where $\mathcal{F}' = \{ f' : f \in \mathcal{F} \}$

Proof Definition + Weierstraß Convergence

Let $\{f_n'\} \subseteq \mathcal{F}'$ be a sequence with $f_n \in \mathcal{F}$.

Pick a subsequence $f_{n_k} \overset{l.u.}{\rightrightarrows} f$. By Weierstraß,

$f_{n_k}' \overset{l.u.}{\rightrightarrows} f'$ showing $\mathcal{F}'$ is normal.

Remark

We can define $\mathcal{F}$ uniformly bounded, locally uniformly bounded etc just as before.

Examples

[1] $\mathcal{F} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic, $f = \sum_{k=1}^{\infty} a_k z^k,\ |a_k| \leq k \}$

locally uniformly bounded.

Indeed, since all compacts $K \subseteq \overline{\Delta}(0,r)$ suffices to work over $\overline{\Delta}(0,r)$. Then

$$|f(z)| \leq \sum_{k=1}^{\infty} |a_k|\,|z|^k \leq \sum_{k=1}^{\infty} k\, r^k = \frac{r}{(1-r)^2} \quad \forall\, |z| \leq r$$

$$\forall\, f \in \mathcal{F}$$

$\Rightarrow \mathcal{F}$ locally uniformly bounded.

[ii] $\mathcal{F}$ family of holomorphic functions in $\mathcal{U}$

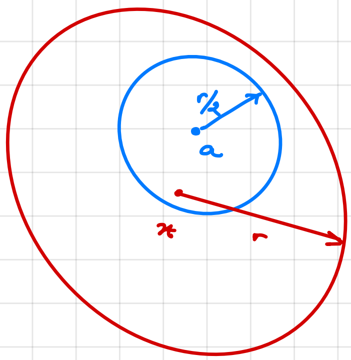
$\mathcal{F}$ locally uniformly bounded. $\Rightarrow$

$\mathcal{F}'$ locally uniformly bounded.

Proof Cauchy's estimates.

Take $z \in \mathcal{U}$. $\Rightarrow \exists\, \Delta(z,r) \subseteq \mathcal{U}$ such that $\forall f \in \mathcal{F}$:

$$|f| \leq M \quad \text{over} \quad \Delta(z,r).$$



We bound $|f'|$ over $\Delta(z, r/2)$.

Let $a \in \Delta(z, r/2)$. By Cauchy's estimate

$$|f'(a)| \leq \frac{\sup |f| \text{ over } \overline{\Delta}(a, r/2)}{r/2} \leq \frac{M}{r/2}.$$

where we used $\overline{\Delta}(a, r/2) \subseteq \Delta(z,r)$.

[iii] On $\Delta(0,1)$, $\mathcal{F} = \{z^n\}_n$ is uniformly bounded

but $\mathcal{F}' = \{nz^{n-1}\}_n$ is not uniformly bounded

Montel Rephrased (Dream Statement)

$\mathcal{F}$ family of holomorphic functions in $\mathcal{U} \subseteq \mathbb{C}$

$\mathcal{F}$ locally uniformly bounded $\iff$ $\mathcal{F}$ normal.

it is customary to say "locally bounded".

Remark Both sides are well behaved under taking derivatives as we noted.

Point #1 All notions we use are local e.g.

local boundedness, local uniform convergence, local equicontinuity (today)

Point #2 Work with families

$\mathcal{F}$ family of continuous or holomorphic functions in $\mathcal{U}$

"Une suite infinie de fonctions analytiques et bornées à l'intérior d'un domaine simplement connexe, admet au moins une function limite à l'intérior de ce domaine"

P. Montel 1907

Paul Montel (1876 – 1975) studied normal families of functions. He proved the above theorem in his thesis in 1907. In 1927 he published a monograph on normal families

Students: Cartan, Dieudonné

Montel's Theorem $\mathcal{F}$ family of holomorphic functions in $\mathcal{U}$

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally bounded.

This fails in real analysis,

$\mathcal{F} = \{\sin nx\}_n$ locally bounded in $\mathbb{R}$ & not normal.

(we can't even arrange pointwise convergence)

Question A What is the correct statement in real analysis i.e. continuous functions?

Remark

This requires the notion of equicontinuity.

There will be several versions.

2. Notions of Equicontinuity

strongest

$\boxed{1}$ $\mathcal{F}$ equicontinuous on $\mathcal{U}$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall |x-y| < \delta \quad \forall f \in \mathcal{F}: \quad |f(x) - f(y)| < \varepsilon.$$

Main Point If $\mathcal{F} = \{f\}$ this says f uniformly continuous.

In general, this says

all $f \in \mathcal{F}$ are uniformly continuous, "uniformly".

that is, the same δ in the definition of uniform continuity works for all $f \in \mathcal{F}$, uniformly.

I Fix $M > 0$. The family

$$\mathcal{F} = \{ f : (0,1) \longrightarrow \mathbb{R},\ |f(x) - f(y)| \leq M\, |x-y| \}$$ equicontinuous.

Suffices to take $\delta = \frac{\varepsilon}{M}$ and note

$$|x-y| < \delta \implies |f(x) - f(y)| \leq M\, |x-y| < \varepsilon \quad \forall f \in \mathcal{F}.$$

II $\mathcal{F} = \left\{ f = \sum_{k=0}^{2024} a_k x^k,\ |a_k| \leq 1 \right\}$ equicontinuous on $[-1,1]$.

$$\left| \frac{f(x) - f(y)}{x-y} \right| = \left| \sum_{k=0}^{2024} a_k \left(x^{k-1} + \dots + y^{k-1} \right) \right|$$

$$\leq \sum_{k=0}^{2024} |a_k| \left(|x|^{k-1} + \dots + |y|^{k-1} \right)$$

$$\leq \sum_{k=0}^{2024} 1 \cdot \underbrace{(1 + \dots + 1)}_{k} = \sum_{k=0}^{2024} k = M$$ & use part I

III $\mathcal{F} = \{ f_n \}$; $f_n(x) = nx$ not equicontinuous in $[0,1]$.

IV See also the Proposition at the end of lecture.

Variations

$\boxed{I}$ equicontinuous (see above)

$\boxed{II}$ equicontinuous at each point (Conway).

$$\forall x \in \mathcal{U} \quad \forall \varepsilon > 0 \quad \exists \Delta(x, \delta) \text{ s.t. } \forall y \in \Delta(x, \delta) \Rightarrow |f(y) - f(x)| < \varepsilon. \quad \forall f \in \mathcal{F}$$

When $\mathcal{F} = \{f\}$ this says f is continuous at each point.

$\boxed{III}$ locally equicontinuous

$$\forall x \quad \exists \Delta_x \subseteq \mathcal{U}, \quad \mathcal{F}/_{\Delta_x} \text{ is equicontinuous}$$

$\boxed{IV}$ equicontinuous on all compacts (Rudin, Ahlfors, us)

$$\forall K \subseteq \mathcal{U} \text{ compact}, \quad \mathcal{F}/_{K} \text{ equicontinuous}$$

$\boxed{ii}$ – $\boxed{iii}$ – $\boxed{iv}$ are equivalent.

$\boxed{iv} \Rightarrow \boxed{iii}$ Just use $K = \overline{\Delta}_*$ where Δ_* is a bounded neighborhood of $*$ in $\mathcal{U}$.

$\boxed{iii} \Rightarrow \boxed{ii}$ clear from definitions

$\boxed{ii} \Rightarrow \boxed{iv}$ requires a compactness argument see Conway. We will not use.

3. Characterization of normality for continuous functions.

Theorem (Arzelà - Ascoli)

$\mathcal{F}$ family of continuous functions

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ is locally equicontinuous & locally bounded.

Theorem (Montel) $\mathcal{F}$ family of holomorphic functions.

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally bounded.

Question B Why is local equicontinuity needed in real analysis?

Question C Why is local equicontinuity NOT needed in complex analysis?

Answer to C

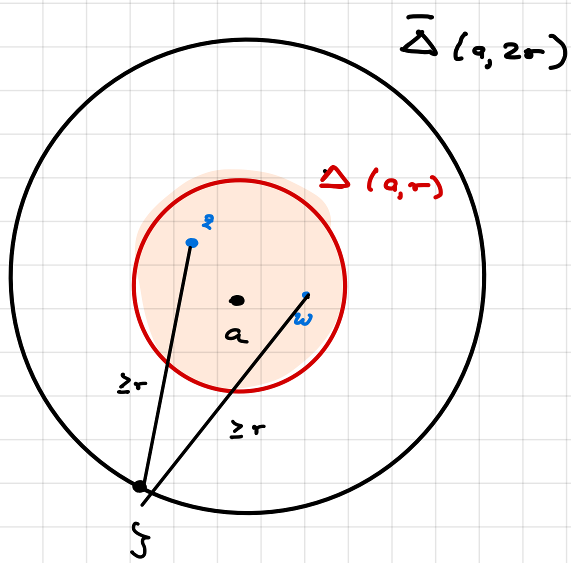
Proposition $\mathcal{F}$ family of holomorphic functions.

$\mathcal{F}$ is locally bounded $\Rightarrow$ $\mathcal{F}$ is locally equicontinuous.

Proof Fix $a \in \mathcal{U}$.

$\Rightarrow \exists\ \bar{\Delta}(a, 2r)$ such that

$\mathcal{F}/_{\bar{\Delta}(a,2r)}$ is bounded by M.



Claim $\mathcal{F}/_{\Delta(a,r)}$ is equicontinuous.

Fix $\varepsilon > 0$. Let $z, w \in \Delta(a,r)$. Take $f \in \mathcal{F}$.

$$|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{|\zeta - a| = 2r} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{|\zeta - a| = 2r} \frac{f(\zeta)}{\zeta - w} d\zeta \right|$$

$\swarrow$ Cauchy's formula

$$= \frac{1}{2\pi} \left| \int_{|\zeta - a| = 2r} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\zeta - a| = 2r} f(\zeta) \cdot \frac{z - w}{\underbrace{(\zeta - z)}_{\geq r} \underbrace{(\zeta - w)}_{\geq r}} d\zeta \right|$$

$$\leq \frac{1}{2\pi} \cdot M \cdot |z-w| \cdot \frac{1}{r^2} \cdot 2\pi \cdot (2r)$$

$$= \frac{2M}{r} \cdot |z-w| = K\,|z-w| \quad \text{for} \quad K = \frac{2M}{r}.$$

The Claim follows by Example $\boxed{c}$ above. or directly,

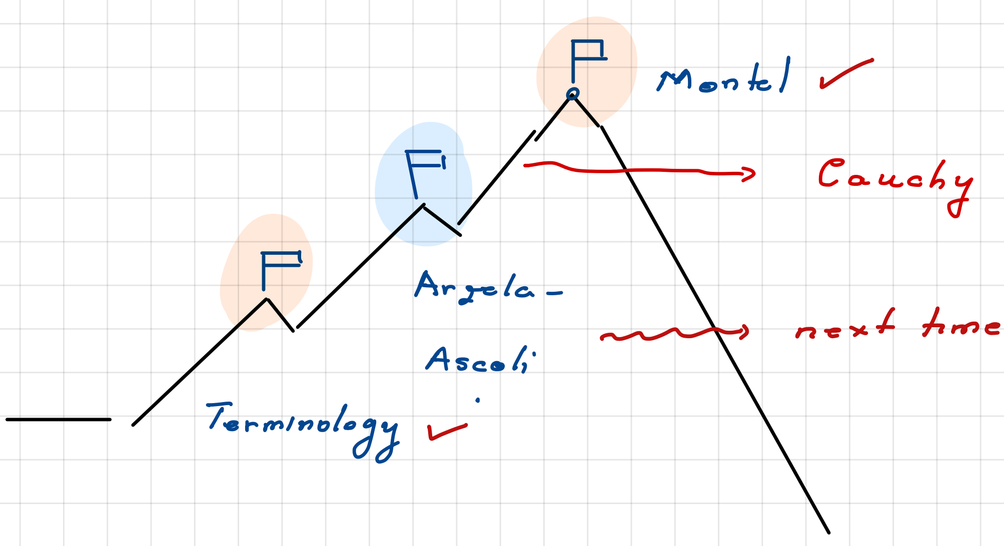
let $\delta = \frac{\varepsilon}{K}$. If $|z-w| < \delta \Rightarrow |f(z) - f(w)| \leq K\,|z-w| < \varepsilon$.

QED

Conclusion

Proposition + Arzelà - Ascoli $\Rightarrow$ Montel

We prove Arzelà - Ascoli next time.



Notation & Preliminaries for Arzelà - Ascoli

$f : U \longrightarrow \mathbb{C}$ continuous, $K \subseteq U$ compact

$$\|f\|_K = \sup_{z \in K.} |f(z)|$$

Note

[I] $\|f+g\|_K \leq \|f\|_K + \|g\|_K$

[II] $f_n \overset{K}{\rightrightarrows} f \iff \|f_n - f\|_K \longrightarrow 0$ as $n \longrightarrow \infty$.

Def f_n is uniformly Cauchy in K if

$\forall\, \varepsilon > 0 \quad \exists\, N \quad \forall\, n, m \geq N, \quad \|f_n - f_m\|_K < \varepsilon.$

Lemma f_n converges uniformly in K

$\iff$ f_n uniformly Cauchy in K.

Proof We will only use "$\Leftarrow$" so we only give its proof.

Fix $\varepsilon > 0 \Rightarrow \exists N$ with $|f_n(z) - f_m(z)| < \varepsilon$ $\forall\, n, m \geq N$.

$\forall\, z \in K$. $(*)$

Thus $\{f_n(z)\}$ is Cauchy for fixed z. Then $\{f_n(z)\}$ converges pointwise to $f(z)$. Make $m \longrightarrow \infty$ in $(*)$ to conclude that

$\forall\, \varepsilon \;\; \exists\, N$ with $|f_n(z) - f(z)| \leq \varepsilon$ $\forall\, n \geq N,\; z \in K$.

Thus $f_n \rightrightarrows f$ in K.