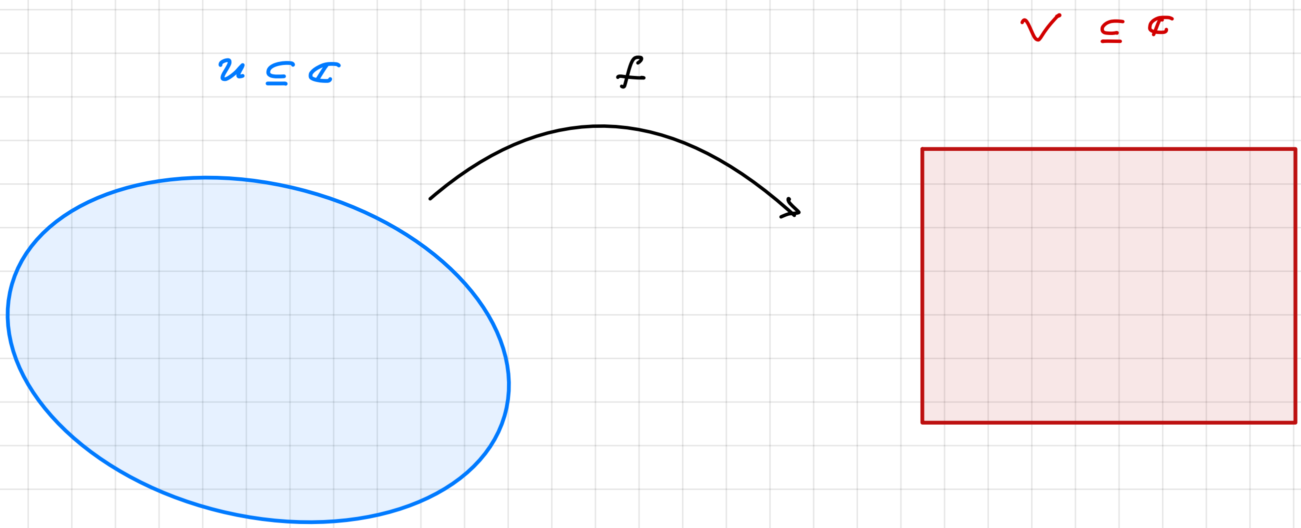
Math 220B — Lecture 9

February 7, 2024

220B Part II — Mapping Theory

The goal is to frame the discussion & formulate guiding questions.



Given $\mathcal{U}, V \subseteq \mathbb{C}$ we wish to study holomorphic

$$f : \mathcal{U} \longrightarrow V.$$

This may be too general. We can ask

[i] f injective

[ii] f finite to one

[iii] f bijective

[iv] f proper ... etc.

We will focus on bijective holomorphic maps.

In Math 220A, we showed

$f : U \to V$ holomorphic & bijective $\Rightarrow$ f^{-1} holomorphic

Biholomorphism = holomorphic + bijective

We focus on biholomorphisms

Question A Given $U, V \subseteq \mathbb{C}$ are U, V biholomorphic?

Remark This has implications in topology & differential geometry. In particular U, V are

homeomorphic, diffeomorphic

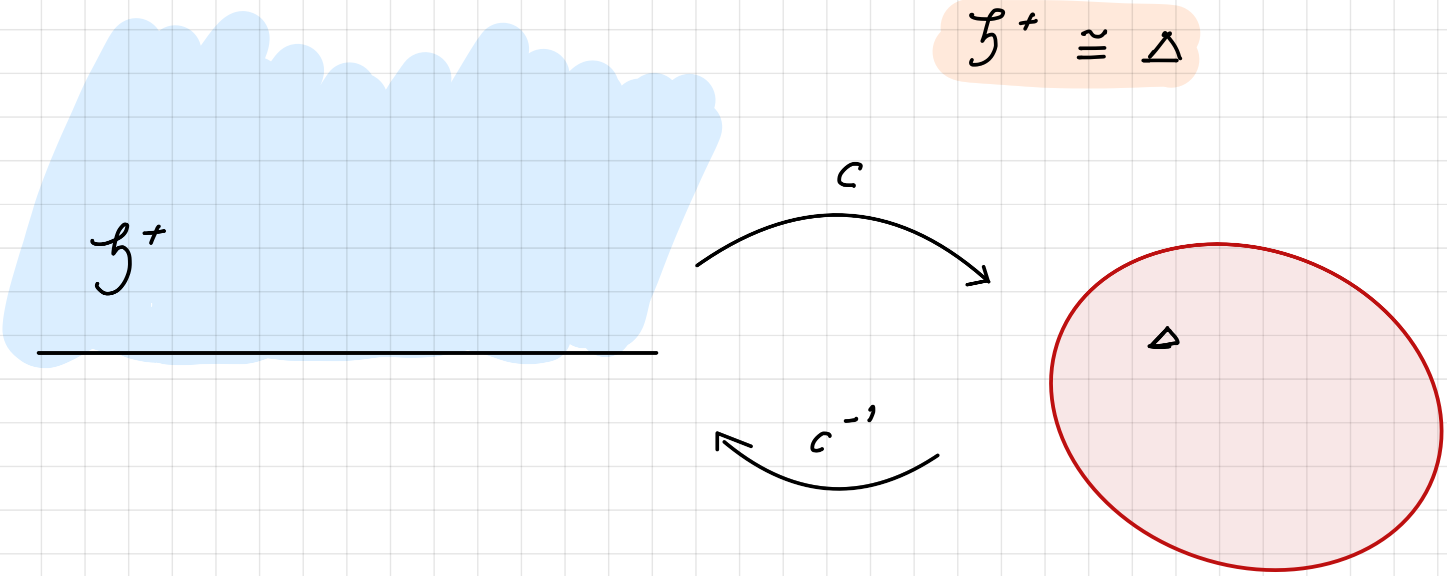
Examples

$\boxed{\mathrm{I}}$ $\mathcal{U} = \mathbb{C}$, $V = \Delta(0,1)$, $\mathcal{U} \overset{\sim}{\not\longrightarrow} V$. This follows by Liouville's theorem.

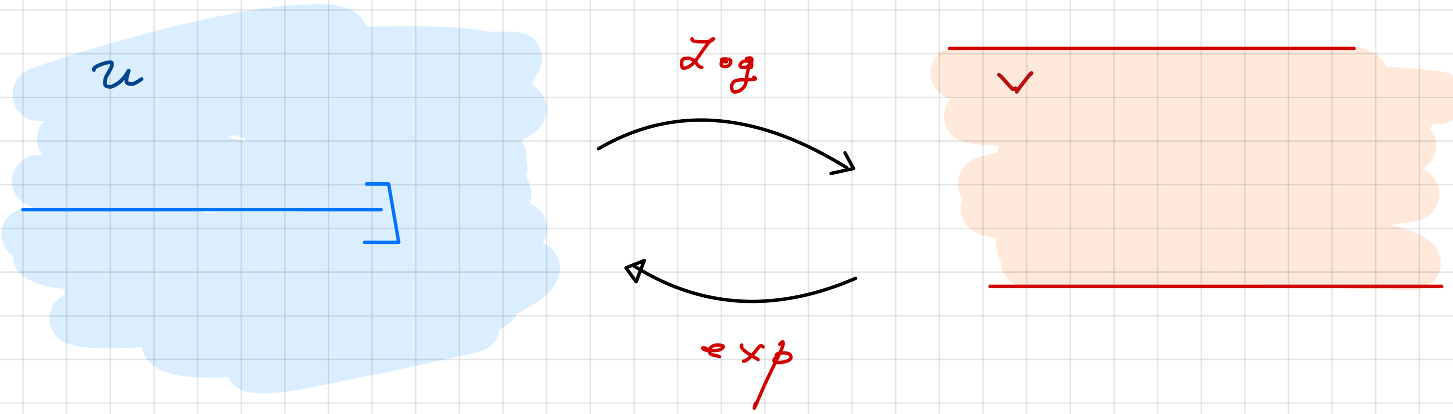
$\boxed{\mathrm{II}}$ $\mathcal{U} = \mathfrak{H}^+$, $V = \Delta$, $c : \mathfrak{H}^+ \longrightarrow \Delta$ Math 220A

Cayley transform: $c(z) = \dfrac{z-i}{z+i}$, $c^{-1}(w) = i \cdot \dfrac{1+w}{1-w}$.

$\mathfrak{H}^+ \cong \Delta$



$\boxed{iii}$ $\mathcal{U} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $V =$ strip $-\pi < \operatorname{Im} z < \pi$



This is Homework 2, Math 220A.

Very Important Theorem (Riemann Mapping)

Given $\mathcal{U}, V \neq \mathbb{C}$, $\mathcal{U}, V$ simply connected $\Rightarrow$ $\mathcal{U}, V$ are biholomorphic.

In particular, if $V = \Delta(0,1)$, then any $\mathcal{U} \neq \mathbb{C}$ simply connected then $\mathcal{U}$ is biholomorphic to $\Delta(0,1)$.

— Riemann's dissertation (1851) sketched a proof.

— Refereed by Gauss

" The whole is a solid work of high quality, not merely fulfilling the requirements usually set for doctoral thesis, but far surpassing them."

— it took the effort of many great minds

Weierstrass, Caratheodory, Hilbert, Schwarz, Koebe, Fejer, Riesz & others to finalize the proof.

Question B

Given $U, V \subseteq \mathbb{C}$ biholomorphic can we construct

$\boxed{i}$ one biholomorphism $U \longrightarrow V$ explicitly?

$\boxed{ii}$ all biholomorphism $U \longrightarrow V$ explicitly?

Special cases of $\boxed{i}$

We saw some specific examples above e.g. the Cayley transform for $\mathcal{H}^+$ and $\Delta(0,1)$.

When $U = V$, Question B [ii] becomes.

Question C

What are all biholomorphisms $f : U \longrightarrow U$?

Remarks

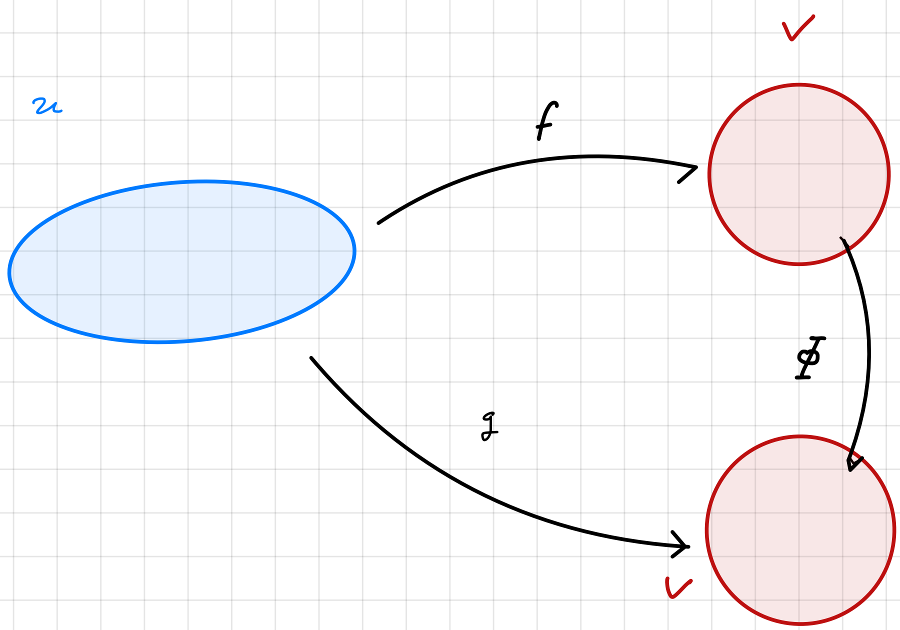
[i] $\mathrm{Aut}(U) = \{ f : U \longrightarrow U : f$ holomorphic & bijective $\}$

is a group.

Indeed $f \in \mathrm{Aut}(U) \Rightarrow f^{-1} \in \mathrm{Aut}(U)$ using that f^{-1} is automatically holomorphic by the above remarks.

[ii] Examples: We can consider this question for $U = \Delta, \mathfrak{H}^+, \mathbb{C}, \Delta^\times, \mathbb{C}^\times$ etc ...

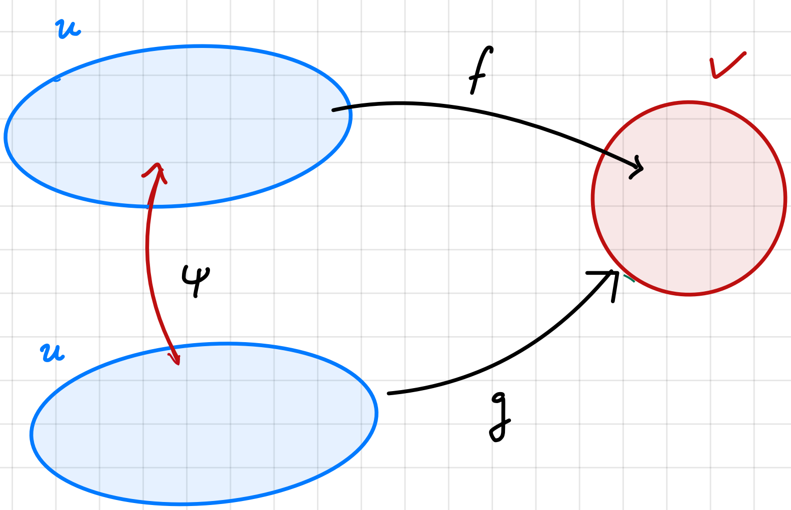
$\boxed{iii}$ If $f, g : U \longrightarrow V$, f = known biholomorphism then any other biholomorphism $g : U \longrightarrow V$ differs from f by automorphisms:



$$g = \Phi \circ f$$

$$\Phi \in Aut(V)$$

Indeed, $\Phi = g \circ f^{-1}$.



In the same fashion

$$g = f \circ \varphi \quad \text{where} \quad \varphi = f^{-1} \circ g$$

and $\varphi \in Aut(U)$.

Thus knowledge of Question C helps with aspects of Question B.

Question D

Is the action of $\mathrm{Aut}(\mathcal{U})$ on $\mathcal{U}$ transitive i.e.

$\forall\, a, b \in \mathcal{U} \quad \exists\, f \in \mathrm{Aut}(\mathcal{U})$ with $f(a) = b$?

Example $\mathcal{U} = \mathbb{C} \cup \{\infty\}$. FLT are automorphisms of $\mathcal{U}$

& action is transitive. (Math 220A)

Question E

Given $a \in \mathcal{U}$, describe $f : \mathcal{U} \longrightarrow \mathcal{U}$ biholomorphism, with $f(a) = a$. $\leadsto$ fixed points.

Many other questions can be asked.

I. Schwarz Lemma – Conway VI. 2.

$\Delta = \Delta(0,1)$

Theorem Given $f : \Delta \longrightarrow \Delta$, $f(0) = 0$ then

[i] $|f'(0)| \leq 1$, and

[ii] $|f(z)| \leq |z|$

If either $|f'(0)| = 1$ or $\exists z_0 \neq 0$ with $|f(z_0)| = |z_0|$ then f is a rotation i.e. $f(z) = e^{i\theta} z$

Proof Let $g(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$. By the removable singularity theorem (Lecture 10, Math 220A), g is holomorphic.

This uses $f(0) = 0$.

Let $0 < r < 1$. Then for $|w| = r$,

$$|g(w)| = \frac{|f(w)|}{|w|} \leq \frac{1}{r} \quad \text{since } \operatorname{Im} f \subseteq \Delta.$$

By maximum modulus principle,

$$\sup_{|w|\leq r} |g(w)| = \sup_{|w|=r} |g(w)| \leq \frac{1}{r}.$$

In particular, for all $|z| < r < 1$, we have

$$|g(z)| \leq \frac{1}{r}$$

Make $r \to 1$ keeping z fixed. Then $|g(z)| \leq 1$. In particular

$|g(0)| = |f'(0)| \leq 1$ & $|f(z)| \leq |z|$.

If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for $z_0 \neq 0$ then either

$|g(0)| = 1$ or $|g(z_0)| = 1$. Since $|g(z)| \leq 1$ $\forall z$ then g must be

constant by MMP again. Thus $g(z) = e^{i\theta}$. $\Rightarrow$ $f(z) = e^{i\theta} z$.

Corollary $f : \Delta \longrightarrow \Delta$ biholomorphism, $f(0) = 0$ then f is a rotation.

Proof Note $f(0) = 0 \Rightarrow f^{-1}(0) = 0$. We apply Schwarz to both f, f^{-1}. We obtain

$|f(z)| \leq z$ and $|f^{-1}(w)| \leq w$. Let $w = f(z)$ to get $|z| \leq |f(z)|$. Therefore $|f(z)| = |z|$ $\forall z \Rightarrow f$ rotation.

II. Automorphisms of the unit disc

$\Delta = \Delta(0,1).$

Question What can we do if we are given

$$f : \Delta \longrightarrow \Delta \quad \text{with} \quad f(0) = a \neq 0, \quad |a| < 1.$$

Key Idea

$$\exists\ \varphi_a : \Delta \longrightarrow \Delta \quad \text{with} \quad \varphi_a(a) = 0.$$

We can then recenter f by considering $\tilde{f} = \varphi_a \circ f$.

$$\begin{tikzcd}
0 \arrow[r, bend left, "f"] \arrow[rr, bend right, "\varphi_a \circ f"'] & a \arrow[r, bend left, "\varphi_a"] & 0
\end{tikzcd}$$

Specifically

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a} z}.$$

Important Properties

[i] $\varphi_a : \Delta \longrightarrow \Delta$, $\varphi_a : \partial\Delta \longrightarrow \partial\Delta$

[ii] $\varphi_a(0) = -a$, $\varphi_a(a) = 0$

[iii] φ_a, φ_{-a} are inverses

[iv] $\varphi_a'(0) = \underbrace{1-|a|^2}_{\text{shrinks} < 1}$, $\varphi_a'(a) = \underbrace{\dfrac{1}{1-|a|^2}}_{\text{expands} > 1}$.

Proof [ii] – [iv] follow by direct calculation.

[i] Note that $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ has pole at $\dfrac{1}{\bar{a}}$ but this is not in $\bar{\Delta}$ since $|a|<1$. Thus φ_a is holomorphic in Δ, continuous in $\bar{\Delta}$. If we show

(*) $|\varphi_a(z)| = 1$ if $|z|=1$, by the maximum modulus, it follows $|\varphi_a(z)|<1$ $\forall$ $|z|<1$ so $\varphi_a : \Delta \longrightarrow \Delta$.

To see (*) we show $|z-a| = |1-\bar{a}z|$ if $|z|=1$.

Note $|1-\bar{a}z| = |1-a\bar{z}|$ conjugation

$$= \left|1-\frac{a}{z}\right| \quad \text{since } z\bar{z} = |z|^2 = 1$$

$$= \frac{|z-a|}{|z|} = |z-a| \quad \text{as needed.}$$

Theorem If $f : \Delta \longrightarrow \Delta$ is biholomorphic then

$$f(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z} \quad \text{for } |a|<1.$$

Proof

$$\begin{tikzcd} 0 \arrow[r, bend left, "\varphi_{-a}"] \arrow[rr, bend right, "f\circ\varphi_{-a}"'] & a \arrow[r, bend left, "f"] & 0 \end{tikzcd}$$

Let a be such that $f(a)=0$. Let

$$\tilde{f} = f \circ \varphi_{-a} \Rightarrow \tilde{f}(0) = 0.$$

Note $\tilde{f}$ is a biholomorphism. Then $\tilde{f}$ is a rotation

$$\Rightarrow \tilde{f}(w) = e^{i\theta} w \Rightarrow f\circ\varphi_{-a}(w) = e^{i\theta} w \Rightarrow f(z) = e^{i\theta}\varphi_a(z).$$

setting $w = \varphi_a(z)$.

Remark We have seen φ_a's in HWK 1.

Blaschke's products f hol. in Δ, continuous in $\overline{\Delta}$,

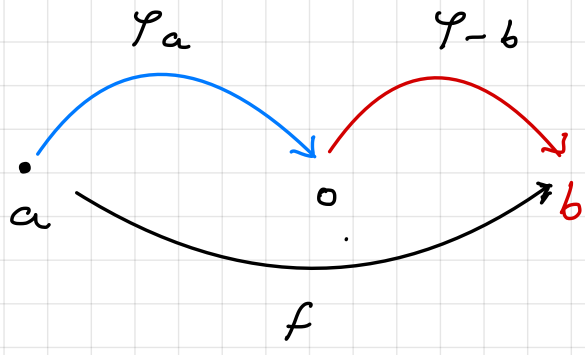
$f: \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$ then

$$f(z) = c\, z^m \prod_{k=1}^{N} \varphi_{a_k} \quad , \quad |c| = 1.$$

$\rightsquigarrow$ zeroes of f.

III. Understanding the action of Aut (Δ) on Δ

Important Remark The action of Aut (Δ) on Δ is transitive $\forall\, a, b \in \Delta \quad \exists\, f \in \mathrm{Aut}\,\Delta, \quad f(a) = b$.



Note $f = \varphi_{-b} \circ \varphi_a$ is an automorphism and

$$f(a) = \varphi_{-b}\, \varphi_a(a) = \varphi_{-b}(0) = b.$$