Math 220B - Winter 2021 - Midterm Exam

Name: ____________________________________

Student ID: ________________________________

Instructions:

There are 5 questions which are worth 50 points.

You may not use any books, notes or internet. If you use a homework problem you will need to reprove it.

There is a 15 minute buffer period to upload your exam in gradescope.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
Total		50

Problem 1. [*10 points; 5, 5.*]

(i) Give an example of an entire function with simple zeroes only at $z = \sqrt{n}$ for each $n \in \mathbb{Z}_{\geq 0}$, and no other zeroes.

(ii) Give an example of a meromorphic function function in $\mathbb{C}$ with poles only at $z = -\sqrt{n}$ and principal parts $\frac{1}{z+\sqrt{n}}$, for $n \in \mathbb{Z}_{\geq 0}$.

Problem 2. [*10 points.*]

Consider $\{a_n\}, \{b_n\}$ two sequences of complex numbers without common terms, such that

$$\sum_{n=1}^{\infty} |a_n - b_n| < \infty$$

and $b_n \to \infty$ as $n \to \infty$. Show that the product

$$f(z) = \prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n}$$

defines a holomorphic function in the open set $\mathbb{C} \setminus \{b_1, b_2, \ldots\}$, and determine its zeros.

Problem 3. [*10 points.*]

Let $f : \Delta(0,1) \setminus \{0\} \to \mathbb{C}$ be a holomorphic function on the punctured unit disc. Let

$$f_n : \Delta(0,1) \setminus \{0\} \to \mathbb{C}, \quad f_n(z) = f\left(\frac{z}{n}\right).$$

Show that the family $\mathcal{F} = \{f_n : n \geq 1\}$ is normal iff f has a removable singularity at the origin.

Problem 4. [*10 points; 4, 6.*]

Recall the function
$$G(z) = \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right).$$

(i) Show that there exists an entire function h such that
$$\left(z+\frac{1}{2}\right) G(z) G\left(z+\frac{1}{2}\right) = e^{h(z)} G(2z).$$

(ii) Show furthermore that $h(z) = az + b$.

Remark: The constants a, b can be found explicitly, by setting $z = 0$ and $z = 1/2$ and computing the relevant values of G from the values of the Γ-function. You can try it for yourself if you are interested. It's good practice with the Γ-function.

Problem 5. [*10 points.*]

Construct a holomorphic function $f : \Delta(1,3) \to \Delta(0,2)$ such that

(i) f extends to a continuous function $f : \overline{\Delta}(1,3) \to \overline{\Delta}(0,2)$ such that $|f(z)| = 2$ if $|z-1| = 3$

(ii) f has a zero at $z = 2$ with order 2 and a simple zero at $z = 3$, and no other zeros.

Please fully justify your answer.