Math 220B - Winter 2024 - Midterm Exam

Name: ______________________________

Student ID: ___________________________

**Instructions:**

There are 5 questions which are worth 50 points.

You may not use any books, notes or internet.

If you use a homework problem you will need to reprove it.

| Question | Score | Maximum |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| Total | | 50 |

**Problem 1.** [*10 points; 4, 6.*]

(i) Write down an entire function with simple zeros only at $z = \sqrt[3]{n}$ for $n = 1, 2, 3, \ldots$, and no other zeros.

(ii) Show that there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(\sqrt[3]{n}) = \sqrt{n}$ for all $n = 1, 2, 3, \ldots$.

**Problem 2.** [*10 points; 5, 5.*]

(i) Show that the infinite product

$$f(z) = \prod_{n=2}^{\infty} (1 - nz^n)$$

defines a holomorphic function in the unit disc $\Delta = \Delta(0,1)$.

(ii) Give one example of a holomorphic function $g : \Delta \to \mathbb{C}$ with simple zeros at $\frac{1}{\sqrt[n]{n}}$ for $n = 2, 3, \ldots$, and such that $g(0) = g'(0) = 1$.

**Problem 3.** [*10 points.*]

Let $\Delta = \Delta(0,1)$ be the unit disc. Show that the family

$$\mathcal{F} = \{f : \Delta \to \mathbb{C} \text{ holomorphic, } |f'(z)|(1-|z|^2) + |f(0)| \leq 1\}$$

is normal.

**Problem 4.** [*10 points.*]

Let $\Delta = \Delta(0,1)$ be the unit disc. Let $f : \overline{\Delta} \to \mathbb{C}$ be continuous, $f$ holomorphic in $\Delta$, and such that

(i) $f$ has zeros at $\frac{1}{2}$ and $\frac{1}{3}$ with multiplicity 2, and no other zeros,

(ii) $|f(z)| = 1$ for $|z| = 1$.

Determine $|f(0)|$. Please justify your answer.

**Problem 5.** [*10 points; 3, 7.*]

(i) For $f : \mathbb{C} \to \mathbb{C}$ entire with a zero of order $m$ at $a$, show that $f'/f$ has a simple pole at $a$ with residue equal to $m$.

(ii) Let $\{a_n\}_{n\geq 1}$ be a sequence of distinct complex numbers with $a_n \to \infty$. Let $\{m_n\}_{n\geq 1}$ be a sequence of positive integers.

Let $g$ be a meromorphic function in $\mathbb{C}$ with simple poles only at $a_n$, and with residues equal to $m_n$ at $a_n$.

Show that there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ such that $h'/h = g$.