Math 220B Final Exam Review

To review, we list below the *Main Topics* covered in this class (this is not a comprehensive list):

(1) Products of holomorphic functions. Convergence, zeroes, logarithmic derivatives.
(2) Weierstraß elementary factors. Weierstraß factorization. Weierstraß problem.
(3) Mittag-Leffler problem in $\mathbb{C}$. Examples.
(4) Factorization of the sine function. The Gamma function.
(5) Normal families. Montel's theorem.
(6) Schwarz's lemma. Automorphisms of the disc. Schwarz-Pick.
(7) Riemann Mapping Theorem.
(8) Schwarz Reflection Principle.
(9) Runge's Theorem. Polynomial and rational approximation. Simple connectivity.

Additional Practice Problems

Please review the course material and the homework problems. In case you need more practice problems, a list is below. There's no need to solve them all before the final; they're here just in case you think you need more practice

1. (*Qualifying Exam 2019.*) Show that there are no bijective holomorphic maps $f : \{0 < |z| < 1\} \to \{1 < |z| < 2\}$.

2. (*Conway, Chapter VI.2.6, page 133.*) Assume f is holomorphic in a region containing $\overline{\Delta} = \overline{\Delta}(0, 1)$, and that $|f(z)| = 1$ if $|z| = 1$. Assume that f has a simple zero at $z = \frac{1}{4}(1 + i)$ and a double zero at $z = \frac{1}{2}$. Can $f(0) = \frac{1}{2}$?

3. (*Conway, Chapter VI.2.8, page 133.*) Is there a holomorphic function $f : \Delta \to \Delta$ such that $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? Is it unique?

4. (*Qualifying Exam 2017.*) Let $f : \mathfrak{h}^+ \to \mathbb{C}$ such that $f(i) = 0$ and $|f(z)| < 1$ for all $z \in \mathfrak{h}^+$. What is the maximum value of $|f(2i)|$?

5. Let f be a holomorphic map between the strip $S = \{-1 < \text{Re } z < 1\}$ and the unit disc $\Delta(0, 1)$ such that $f(0) = 0$. What is the maximum value of $|f'(0)|$?

6. Let $A = \{2 < |z| < 3\}$ and $f(z) = \frac{e^z}{z}$. Can the function f be approximated locally uniformly in A by polynomials? Can it be approximated locally uniformly in A by rational functions with poles only at 1 and 4? By rational functions with poles only at 4?

7. (*Conway, Chapter VII.6.1, page 166.*) Show that

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

8. Show that $f(z) = -\cos z$ determines a biholomorphism between the half infinite strip $\{z = x + iy : 0 < x < \pi, y > 0\}$ and the upper half plane.

9. (*Qualifying Exam 2017.*) Construct a meromorphic function with simple poles at $z = n$ and residues equal to $n\sqrt{n}$ for $n = 1, 2, 3 \ldots$.

10. (*Qualifying Exam 2008.*) Prove that there exist a sequence R_n of rational functions whose finite poles are only at $\frac{3}{2}$ such that

$$\lim_{n\to\infty} R_n(z) = 1 \quad \text{for } |z| = 1, \quad \lim_{n\to\infty} R_n(z) = 2 \quad \text{for } 2 \le |z| \le 3.$$

11. (*Qualifying Exam 2020.*) Let $G \neq \mathbb{C}$ be a connected set such that $\widehat{\mathbb{C}} \setminus G$ is connected. Show that if $f : G \to G$ is holomorphic and admits 2 fixed points then f is the identity.

12. (*Conway, Chapter VIII.3.5, page 209.*) Assume that a_n is an infinite sequence such that $|a_n| \to \infty$ and let $A_n \in \mathbb{C}$. Show that there exists an entire function f such that $f(a_n) = A_n$.

13. (*Qualifying Exam 2009.*) Assume that α_n is an infinite sequence such that $|\alpha_n| \to \infty$ and let β_n be complex numbers. Show that there exists an entire function such that $f(\alpha_n) = \beta_n$ with multiplicity 2, that is $f - \beta_n$ has a zero of order 2 at α_n.

14. (*Qualifying Exam 2016.*) Let $a_k = 1 - \frac{1}{k^2}$ for $k \ge 1$. Let $f_n(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - z a_k}$.

(i) Show that f_k converges to a holomorphic function $f : \Delta(0,1) \to \Delta(0,1)$.

(ii) Show that there does not exist an open set $U \subset \mathbb{C}$ and a holomorphic function $g : U \to \mathbb{C}$ such that $\overline{\Delta}(0,1) \subset U$ and $f(z) = g(z)$ for $z \in \Delta$.

15. Let f be a nonconstant entire function. Show that there exists $|z| > 2001$ such that $f(z)$ is positive real.

16. Show that there exist polynomials $p_n(z)$ such that $p_n(z) \to 1$ if $\text{Re } z > 0$ and $p_n(z) \to -1$ if $\text{Re } z < 0$.