Math 259 - Homework 2. Due February 3.

1. *Sheafification.* Check the details of the sheafification lemma stated and partially proved in class. Specifically, let $\mathcal{A} \to X$ be a presheaf on a topological space X. Show that

(i) There exists a sheaf $\mathcal{A}^{\mathrm{sh}}$ together with a morphism $\iota : \mathcal{A} \to \mathcal{A}^{\mathrm{sh}}$ such that for all presheaf morphisms $f : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is a sheaf, there exists a unique sheaf morphism $f^{\mathrm{sh}} : \mathcal{A}^{\mathrm{sh}} \to \mathcal{B}$ such that $f^{\mathrm{sh}} \circ \iota = f$.

(ii) Show that the stalks of $\mathcal{A}$ and $\mathcal{A}^{\mathrm{sh}}$ coincide.

(iii) Show that if $\mathcal{A}$ is a sheaf, then $\mathcal{A} = \mathcal{A}^{\mathrm{sh}}$.

2. *Real and holomorphic Poincaré lemma.*

(i) Briefly sketch the proof of the real Poincaré lemma, and briefly explain that if X is a real manifold, then

$$0 \to \mathbb{R} \xrightarrow{\iota} \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{A}^n \to 0$$

is an exact sequence of sheaves, where $\mathcal{A}^k$ denotes the sheaf of smooth k-forms.

Hint: If $\omega = \sum_I \omega_I dx_I$ is a closed k-form defined over a star-shaped domain in $\mathbb{R}^n$, show that ω is exact, i.e. $\omega = d\eta$, by considering the $(k-1)$-form

$$\eta = \sum_I \sum_{j=1}^{k} \left(\int_0^1 t^{k-1} \omega_I(tx)\, dt \right) (-1)^{j-1} x_{i_j} dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k}.$$

(ii) Prove the holomorphic Poincaré lemma. That is, show that if $\Delta \subset \mathbb{C}^n$ is a polydisk and ω is a holomorphic differential k-form over Δ with $d\omega = 0$, then ω is an exact form $\omega = d\eta$, for some holomorphic $(k-1)$-form η over Δ.

(iii) Deduce that if X is a complex manifold of dimension n, then we have an exact sequence of sheaves

$$0 \to \mathbb{C} \xrightarrow{\iota} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \to 0,$$

where Ω^k denotes the sheaf of holomorphic k-forms.