MIDTERM II SOLUTIONS

1. *Consider the points P, Q, R with coordinates $(2,1,0)$, $(0,1,3)$ and $(1,0,1)$ respectively.*

(i) *Find the area of the parallelogram spanned by the vectors $\vec{PQ}$ and $\vec{PR}$.*
(ii) *Find the equation of the plane through P, Q, R.*

Answer: (i) We compute

$$\vec{PQ} = (-2,0,3), \vec{PR} = (-1,-1,1).$$

The cross product equals

$$\vec{PQ} \times \vec{PR} = (3,-1,2).$$

The area of the parallelogram is

$$||\vec{PQ} \times \vec{PR}|| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}.$$

(ii) The plane has equation

$$3(x-2) - (y-1) + 2z = 0 \implies 3x - y + 2z = 5.$$

□

2.*Find the second order Taylor polynomial near $(1,-1)$ for the function*

$$f(x,y) = x^3 y.$$

Answer: We compute

$$f(1,-1) = -1,$$
$$f_x = 3x^2 y \implies f_x(1,-1) = -3$$
$$f_y = x^3 \implies f_y(1,-1) = 1,$$
$$f_{xx} = 6xy \implies f_{xx}(1,-1) = -6,$$
$$f_{xy} = 3x^2 \implies f_{xy}(1,-1) = 3,$$
$$f_{yy} = 0.$$

The Taylor polynomial is

$$P_2 = -1 - 3(x-1) + (y+1) - 3(x-1)^2 + 3(x-1)(y+1).$$

□

3. *Consider the function*

$$f(x,y) = x^4 y^3.$$

(i) *Write down the equation of the tangent plane at the graph of the function at the point $(1,1,1)$.*
(ii) *Write down an expression for the change, Δz, in $z = f(x,y)$ depending on Δx and Δy, the change in x and y, respectively, near the point $x = y = 1$. Is the function $f(x,y)$ more sensitive to a change in x or to a change in y?*

(iii) *Using your answer to (ii), find the approximate value of* $f(1.01, 1.02)$.

Answer: (i) We compute

$$f_x = 4x^3y^3 \implies f_x(1,1) = 4$$

$$f_y = 3x^4y^2 \implies f_y(1,1) = 3.$$

The tangent plane is

$$z - 1 = 4(x-1) + 3(y-1) \implies 4x + 3y - z = 6.$$

(ii)

$$\Delta z = 4\Delta x + 3\Delta y.$$

The function is more sensitive to a change in x because the x derivative at $(1,1)$ is higher.

(iii) We have

$$\Delta x = 1.01 - 1 = .01, \Delta y = 1.02 - 1 = .02,$$

hence

$$\Delta z = 4(.01) + 3(.02) = .1.$$

This gives

$$z(1.01, 1.02) = z(1,1) + \Delta z = 1.1 \implies f(1.01, 1.02) \approx 1.1.$$

□

4. *Consider the function* $f(x,y) = xe^{x+y}$ *and the point* $P = (2, -2)$.

(i) *Find the gradient of* f *at* P.
(ii) *Find the directional derivative of* f *at* P *in the direction* $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$.
(iii) *What is the direction of steepest increase for the function* f *at* P*? Express your answer as a unit vector.*

Answer: (i)

$$f_x = e^{x+y} + xe^{x+y} \implies f_x(2,-2) = e^0 + 2e^0 = 3,$$

$$f_y = xe^{x+y} \implies f_y(2,-2) = 2e^0 = 2.$$

We have

$$\nabla f(P) = (3, 2).$$

(ii)

$$f_{\mathbf{u}}(P) = \nabla f(P) \cdot \mathbf{u} = (3,2) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

(iii) The direction of steepest increase is given by the gradient. Since we want a unit vector, we divide by the length

$$\mathbf{v} = \frac{(3,2)}{||(3,2)||} = \frac{(3,2)}{\sqrt{13}}.$$

□

5. *Consider the function*

$$w = \sin(xy)$$

where

$$x = \frac{1}{v}, \quad y = u^2 v.$$

Using the chain rule, calculate the derivatives

$$\frac{\partial w}{\partial u} \quad \textit{and} \quad \frac{\partial w}{\partial v}.$$

Please express your answer in simplest form.

Answer: We compute

$$\frac{\partial w}{\partial x} = y\cos(xy) = u^2 v\cos(u^2), \qquad \frac{\partial w}{\partial y} = x\cos(xy) = \frac{1}{v}\cos(u^2)$$

$$\frac{\partial x}{\partial u} = 0, \qquad \frac{\partial x}{\partial v} = -\frac{1}{v^2}$$

$$\frac{\partial y}{\partial u} = 2uv, \qquad \frac{\partial y}{\partial v} = u^2.$$

Then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = u^2 v\cos(u^2)\cdot 0 + \frac{1}{v}\cos(u^2)\cdot(2uv) = 2u\cos(u^2)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} = u^2 v\cos(u^2)\cdot\frac{-1}{v^2} + \frac{1}{v}\cos(u^2)\cdot u^2 = 0.$$

□