Math 31AH - Fall 2013 - Midterm I

Name: ____________________________________

Student ID: ______________________________

Instructions:

Please print your name and student ID.
During the test, you may not use books, calculators or telephones.
Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 5 questions + extra credit. They are worth 50 + 5 points. You have 50 minutes to complete the test.

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<th>Question</th>
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Problem 1. [13 points.]

Consider the matrix

\[ A = \begin{bmatrix} -2 & 2 & -3 & 1 \\ -4 & 5 & -9 & 1 \\ 2 & -5 & 12 & 2 \end{bmatrix}. \]

(i) [4] Give a basis for the column space \( C(A) \). What is the rank of \( A \)?
(ii) [3] Give a basis for the null space of $A$. What is the nullity of $A$?

(iii) [3] Show that the columns $c_1, c_2, c_3, c_4$ of $A$ are linearly dependent by exhibiting explicit relations between them.
(iv) [3] Write down the general solution to the following system of equations

\[ Ax = \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix} . \]
Problem 2. [8 points.]

Find the inverse of the following matrix

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
-2 & -3 & 0
\end{bmatrix}.
\]
Problem 3. [11 points.]

Consider the points $A, B, C$ in $\mathbb{R}^3$, with coordinates $(1, -1, 3)$, $(2, -1, 2)$ and $(1, 1, -1)$ respectively.

(i) [3] Find the angle between the vectors $\vec{AB}$ and $\vec{AC}$.

(ii) [3] Using cross product, find the area of the triangle $ABC$. 

(iii) [2] Write down a vector perpendicular to the plane containing the points $A, B, C$.

(iv) [3] Using (iii), write down the equation of the plane passing through $A, B, C$. Your answer should take the form

$$ax + by + cz = d$$

for suitable constants $a, b, c, d$. 
Problem 4. [10 points.]

Circle TRUE (T) or FALSE (F). No explanation is needed.

(T) (F) The columns of $4 \times 4$ matrix $A$ form a basis of $\mathbb{R}^4$ if and only if $A$ is invertible.

(T) (F) The rows of $4 \times 4$ matrix $A$ form a basis of $\mathbb{R}^4$ if and only if $A$ is invertible.

(T) (F) Let $V$ be a 3-dimensional subspace of $\mathbb{R}^6$. Any five vectors $v_1, v_2, v_3, v_4, v_5$ of $V$ are linearly dependent.

(T) (F) For a $4 \times 3$ matrix $A$, there is always a free variable.

(T) (F) The rank of a $4 \times 6$ matrix cannot be equal to 5.

(T) (F) For any $3 \times 5$ matrix, the null space is at least 2 dimensional.

(T) (F) Vectors $u, v, w$ of $\mathbb{R}^3$ are linearly dependent if and only if

$$u \cdot (v \times w) = 0.$$ 

(T) (F) Every system of 2 equations with 3 unknowns has a solution.

(T) (F) The set $\{(m, n) : m, n \text{ are integers}\}$ is a subspace of $\mathbb{R}^2$.

(T) (F) If $A$ is a $5 \times 8$ matrix, there exists a vector $b \in \mathbb{R}^5$ such that the equation $Ax = b$ has a unique solution.
Problem 5. [8 points.]

Fix a basis $u_1, \ldots, u_k$ of $\mathbb{R}^k$, and let $A$ be a $k \times k$ matrix. Show the vectors $Au_1, \ldots, Au_k$
form a basis of $\mathbb{R}^k$ if and only if $A$ is invertible.
Extra credit. [5 points.]

Show that if $V$ and $W$ are subspaces of $\mathbb{R}^n$, then $V \cap W$ is also a subspace of $\mathbb{R}^n$. 