Math 31AH - Fall 2010 - Midterm I

Name: ______________________________

Student ID: _________________________

Instructions:

Please print your name and student ID.

During the test, you may not use books or notes.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 6 questions which are worth 45 points, and a bonus question. You have 50 minutes to complete the test.

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Problem 1. [4 points.]

Complete the following sentences:

(i) [2] $V$ is a subspace of $\mathbb{R}^n$ if ... $V$ is closed under vector addition and multiplication by scalars.

(ii) [2] the vectors $v_1, \ldots, v_k$ are linearly independent if ... for any constants $c_1, \ldots, c_k$, not all zero, the linear combination

\[ c_1 v_1 + \ldots + c_k v_k \]

is not zero.
Problem 2. [16 points.]

Consider the matrix

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 5 & 7 \\
-1 & 0 & -1 & -2
\end{pmatrix}
\]

(i) [6] Give the equation for \( C(A) \) as a subspace of \( \mathbb{R}^3 \).

We row-reduce the augmented matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & b_1 \\
2 & 3 & 5 & 7 & b_2 \\
-1 & 0 & -1 & -2 & b_3
\end{pmatrix}
\]

\[
\rightarrow \begin{pmatrix}
1 & 0 & 1 & 2 & 2b_2 - 3b_1 \\
0 & 1 & 1 & 1 & -b_2 + 2b_1 \\
0 & 0 & 0 & 0 & 2b_2 + b_3 - 3b_1
\end{pmatrix}
\]

The last row of zeros gives the equation

\[
2b_2 + b_3 - 3b_1 = 0
\]

that must be satisfied by all vectors \( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \) in the column space.

The null space of $A$ is also the null space of the row-reduced matrix

$$rref(A) = \begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$ 

The first two variables $x, y$ are pivots, the last two variables $z, w$ are free. We obtain the system

$$x + z + 2w = 0 \implies x = -z - 2w$$
$$y + z + w = 0 \implies y = -z - w.$$

We conclude

$$\begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} = z \begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix} + w \begin{bmatrix}
0 \\
-1 \\
0 \\
1
\end{bmatrix}.$$ 

Therefore $N(A)$ has the basis

$$N(A) = \text{span} \left\{ \begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-2 \\
-1 \\
0 \\
1
\end{bmatrix} \right\}.$$ 

(iii) [3] Show that the columns $c_1, c_2, c_3, c_4$ of $A$ are linearly dependent by exhibiting explicit relations between them.

Each vector in the null space gives a relation between the columns of $A$. For instance, the vector

$$\begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix} \in N(A) \implies -c_1 - c_2 + c_3 = 0$$
and
\[
\begin{bmatrix}
-2 \\ -1 \\ 0 \\ 1
\end{bmatrix} \in N(A) \implies -2c_1 - c_2 + c_4 = 0.
\]

(iv) [3] Give a basis for $C(A)$.

The pivot columns, namely the first and second columns of $A$, give a basis for $C(A)$:

\[
C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.
\]
Problem 3. [7 points]

If the vectors \(\{u, v, w\}\) form a basis of \(\mathbb{R}^3\), show that \(\{u, v - w, u + v - 2w\}\) also form a basis of \(\mathbb{R}^3\).

Assume first that the vectors \(\{u, v - w, u + v - 2w\}\) are linearly dependent. There exist constants \(a, b, c\) not all zero such that

\[
a u + b(v - w) + c(u + v - 2w) = 0.
\]

Rearranging, we obtain

\[
(a + c)u + (b + c)v + (b - 2c)w = 0.
\]

Since \(\{u, v, w\}\) is a basis, the vectors \(\{u, v, w\}\) must be independent. This implies that

\[
a + c = 0, \quad b + c = 0, \quad b - 2c = 0.
\]

Solving for \(a, b, c\) we find

\[
a = b = c = 0
\]

which is impossible by assumption that not all \((a, b, c)\) are zero.

It follows that \(\{u, v - w, u + v - 2w\}\) are three linearly independent vectors in \(\mathbb{R}^3\) hence they must form a basis of \(\mathbb{R}^3\).
Problem 4. [6 points]

Which of the following are subspaces? Simply write “SUBSPACE” or “NOT SUBSPACE”:

(i) [2] The set \( \{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1 \} \)

NOT SUBSPACE

(ii) [2] The set \( \{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : x - 2y + 3z - 4w = 0 \text{ and } x - y - z + w = 0 \} \)

SUBSPACE

(iii) [2] The set of vectors in \( \mathbb{R}^3 \) perpendicular to the vector \( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \).

SUBSPACE
Problem 5. [7 points]

Let $T$ be a $2 \times 3$ such that

\[
T \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad T \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},
\]

Calculate $T \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Note that

\[
\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\]

Therefore,

\[
T \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 2T \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - T \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + T \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.
\]
Problem 6. [5 points]

The vectors $x$ and $y$ have equal length. Show that $x - y$ and $x + y$ are perpendicular.

We know $x$ and $y$ have equal length hence

$$x \cdot x = y \cdot y.$$ 

We calculate the dot product

$$(x - y) \cdot (x + y) = x \cdot x + x \cdot y - y \cdot x - y \cdot y = x \cdot x - y \cdot y = 0.$$ 

Since the dot product is zero, the two vectors $x - y$ and $x + y$ must be perpendicular.
**Extra Credit.** [5 points.]

Assume $A$ is a square $k \times k$ matrix, $B$ is a $k \times (k - 1)$ matrix and $C$ is a $(k - 1) \times k$ matrix. Furthermore, assume that

$$A = BC.$$

Is it true that $N(A)$ always contains a non-zero vector?

Yes! To explain this fact, note that $C$ has fewer rows than columns. Then $C$ must have free variables. This in turn means that the null space $N(C)$ contains a non-zero vector $x$:

$$Cx = 0.$$

We claim that $x$ is also in the null space of $A$. Indeed,

$$Ax = BCx = B \cdot 0 = 0.$$

This shows that $N(A)$ contains a nonzero vector.