RECOMMENDED REVIEW PROBLEMS FOR FINAL EXAM

First, review all the homework problems. In addition, you may want to solve the following problems. Solutions will not be provided, but to make up for it, I will hold office hours during exam week to answer any questions which may arise.

Topics for the final exam:

1. Integration in rectangular coordinates. Limits of integration.
2. Changing order of integration.
3. Polar, cylindrical, spherical coordinates.
4. The change of variables formula.
7. Integration of forms over general oriented or parametrized manifolds.
9. Green’s theorem.
10. Gauss’ theorem.
11. The classical Stokes’ theorem.
12. The generalized Stokes’ theorem.
13. Potentials.

1.A. Find the area of the region between by the curves $xy = 1$, $y = x^2$ and $y^2 = 8x$.

1.B. Find the volume bounded by the paraboloid $x = y^2 + z^2$ and the plane $x = 2y + 3$.

2. Change the order of integration and evaluate

$$
\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3) \, dy \, dx.
$$

3.A. Consider a solid hemisphere of radius $a$ with center at the origin placed above the $(xy)$-plane. Find the average value of the distance of a point in the solid hemisphere from the $(xy)$ plane.

3.B. Using cylindrical coordinates, determine the volume of the solid bounded above by the sphere of radius 2 centered at the origin, and below by the paraboloid $z = x^2 + y^2$.

4. Consider a plate $R$ bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$ and $xy^2 = 4$, and assume that the density function is $\delta(x, y) = y$. Find the first coordinate $\bar{x}$ of the center of mass. This means, calculate the integral

$$
\bar{x} = \frac{\int \int_R x \delta \, dA}{\int \int_R \delta \, dA}.
$$

6.A. Solve exercise 6.7 page 679.


7.A. Consider the manifold \( M \subset \mathbb{R}^4 \) given by the graph
\[
x_1 = x_2 + x_3^2 + x_4^2,
\]
and oriented using the form
\[
\Omega = dx_2 \wedge dx_3 \wedge dx_4.
\]
Find the integral of the form
\[
\phi = x_1 x_2 \, dx_1 \wedge dx_2 \wedge dx_3
\]
over \( M \).

7.B. A certain torus \( S \subset \mathbb{R}^4 \) admits the parametrization \( \gamma : [0, 2\pi) \times [0, 2\pi) \rightarrow S \) given by
\[
\gamma \left( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) = \begin{bmatrix} p \sin \alpha - q \cos \alpha \\ q \sin \alpha + p \cos \alpha \\ r \sin \beta + s \cos \beta \\ r \cos \beta - s \sin \beta \end{bmatrix}.
\]
Find the surface area of the torus in terms of \( p, q, r, s \).

8.A. Show that
\[
\nabla \times (f \mathbf{G}) = f \nabla \times \mathbf{G} + \nabla f \times \mathbf{G},
\]
for any field \( \mathbf{G} \) and any function \( f \) in \( \mathbb{R}^3 \).


9.A. Let \( R \) be part of the region \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \) contained in the first quadrant. Consider the field
\[
\mathbf{F} = \langle -xy^2, y^3 \rangle.
\]
Verify that Green’s theorem is satisfied for the region \( R \) and the field \( \mathbf{F} \). That is, evaluate the two integrals which are the subject of Green’s theorem, and check that they are equal.

9.B. Let \( R \) be the region bounded by the lines \( y = x, \ x = 2, \ y = 0 \) and the hyperbola \( xy = 1 \). Consider the field
\[
\mathbf{F} = x^4 \mathbf{i} + (-4x^3y + y^2) \mathbf{j}.
\]
Use Green’s theorem in normal form to find the outwards flux of the field \( \mathbf{F} \) over the boundary of the region \( R \).

10.A. Consider the field \( \mathbf{F} = \langle -y, x, z \rangle \)
10.B. Let \( S \) be the surface formed by the part of the graph of the paraboloid \( z = x^2 + y^2 \) lying below the plane \( z = 1 \). Let
\[
\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (1 - 2z)\mathbf{k}.
\]
(i) Calculate directly the flux of \( \mathbf{F} \) across \( S \), taking the outward direction as the one for which the flux is positive.
(ii) Repeat the calculation using the divergence theorem.

11.A. Let \( S \subset \mathbb{R}^3 \) be a surface given by an equation \( f(x, z) = 0 \) in \( x \) and \( z \) alone. Show that if \( C \) is any simple closed curve on \( S \), and
\[
\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}
\]
then
\[
\int_C W\mathbf{F} = 0.
\]

11.B. Consider the field
\[
\mathbf{F} = (y, z, x).
\]
Let \( C \) be the base circle of a half sphere of radius 1 centered at the origin such that \( C \) lies in the \((x, y)\) coordinate plane.
(i) Parametrize \( C \) to evaluate the line integral
\[
\int_C W\mathbf{F}.
\]
(ii) Compute the curl \( \nabla \times \mathbf{F} \).
(iii) View \( C \) as the boundary of the unit disc \( R \) contained in the \((x, y)\) plane. Check that Stokes’ theorem holds. That is, compute both sides of the equation
\[
\int_C W\mathbf{F} = \int \int_R \Phi \nabla \times \mathbf{F}
\]
and check they are equal.
(iv) Check that Stokes’ theorem holds when \( R \) is now the paraboloid \( z = 1 - x^2 - y^2 \geq 0 \).


13.A.

(i) Determine the constants $a, b, c$ for which the field

$$
\mathbf{F} = (axy + z^3)i + (x^2 + byz)j + (y^2 + cxz^2 + 1)k
$$

for which the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

(ii) Find a potential function for $\mathbf{F}$.

(iii) Find the equation of a surface $S \subset \mathbb{R}^3$ with the property that for any point $P$ on the surface

$$
\int_{(1,1,1)}^P \mathbf{F} \cdot d\mathbf{r} = 0.
$$

13.B. Let $C$ be the curve parametrized as

$$x(t) = 2 \cos t, y(t) = 3 \sin t, \quad 0 \leq t \leq \pi.$$

Find the value of the line integral

$$
\int_C (\cos x + y^2 \sin x)dx + (\sin y - 2y \cos x)dy
$$

by any method you wish.