REVIEW PROBLEMS FOR MIDTERM I

First, review all the homework problems. In addition, you may want to solve the following problems. Solutions for the textbook problems can be found in the solution manual.


**A.** Evaluate by changing the order of integration
\[
\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx \, dy.
\]

**B.** Find the area of the region in the first quadrant bounded by the hyperbola \(xy = 1\) and the parabolas \(x = y^2\) and \(x = 8y^2\). Express the answer in the simplest possible form.

**C.** Find the volume of the solid bounded by the cylinders
\[z = 1 - y^2, \ z = y^2 - 1\]
and the planes \(x = 0\) and \(x + z = 1\).

**D.** Find the volume of the solid in \(\mathbb{R}^4\) given by
\[x^2 + y^2 + z^2 + w^4 \leq 1\]
using Cavalieri’s principle. The answer should be in terms of the beta function.

(2) Polar, cylindrical, spherical coordinates.

**A.** Using cylindrical coordinates, find the mass of the solid in the upper half space \(z \geq 0\) with density \(\rho = z\) bounded by the sphere \(x^2 + y^2 + z^2 = 2\) and the cone \(z^2 = x^2 + y^2\).

**B.** Using spherical coordinates, find the volume of the cap of the spherical cap that lies between the sphere
\[x^2 + y^2 + z^2 \leq 2\]
and above the plane \(z = 1\).

**C.** From the textbook, solve problem 4.10.9

(3) Arbitrary changes of variables.

**A.** Calculate the mass of a plate with density \(\delta(x, y) = xy\) contained between the ellipses \(x^2 + \frac{y^2}{4} = 2, \ x^2 + \frac{y^2}{4} = 4\) and the hyperbolas \(x^2 - y^2 = 1\) and \(x^2 - y^2 = 2\).

**B.** Consider the region \(D\) bounded by the curves
\[1 \leq x^{30} + y^{60} \leq 2, \ y^2 \leq x \leq 2y^2.\]
Integrate the function \(f(x, y) = \frac{x}{y^3}\) over \(D\).

**C.** From the textbook, solve problem 4.10.19.
(4) **Integrable functions. Measure theory.**

As usual, this topic is a bit more difficult than the rest.

**A.** Using the definition with Riemann sums, calculate the integral of the function \( f(x) = e^x \) over the interval \([0, 1]\).

*Hint: To calculate the Riemann sums, you may need to remember the geometric series*

\[
1 + q + q^2 + \ldots + q^m = \frac{1 - q^{m+1}}{1 - q}.
\]

**B.** Show that if \( f, g : [0, 1] \to [0, 1] \) are integrable functions, and if \( f \) is continuous, then \( f \circ g \) is integrable.

*Hint: The integrability theorem. Where is \( f \circ g \) discontinuous?*

**C.** Solve problem 4.3.2.

**D.** Show that if \( X_1, \ldots, X_n \ldots \) is an infinite sequence of sets, all of measure zero, then their union \( X_1 \cup X_2 \cup \ldots \cup X_n \cup \ldots \) also has measure zero.

*Hint: Generalize the proof given in class showing that any infinite sequence of numbers has measure zero.*