SOLUTIONS TO PROBLEMS WHICH ARE NOT IN THE SOLUTIONS MANUAL.

4.

(i) We have
\[ f_x = 2xy^4 + y^2 \ln(2x - y) + xy^2 \cdot \frac{2}{2x - y} \implies f_x(1, 1) = 4. \]

Next,
\[ f_y = 4x^2y^3 + 2xy \ln(2x - y) + xy^2 \cdot \frac{-1}{2x - y} \implies f_y(1, 1) = 3. \]

Thus \( \nabla f(1, 1) = (4, 3) \). The direction of steepest increase is \((4, 3)\).

(ii) We have
\[ D_x f(P) = \nabla f \cdot \vec{v} = (4, 3) \cdot (1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{\sqrt{2}}. \]

(iii) We have \( f(P) = 1 \) hence
\[ z - 1 = 4(x - 1) + 3(y - 1) \implies z = 4x + 3y - 6. \]

(iv) We set \( g(x, y, z) = z^2x^3 - f(x, y) \). We find
\[ g_x = 3x^2z^2 - f_x \implies g_x(1, 1, 1) = 3 - 4 = -1 \]
\[ g_y = -f_y \implies g_y(1, 1, 1) = -3 \]
\[ g_z = 2zx^3 \implies g_z(1, 1, 1) = 2. \]

The tangent plane is
\[ -(x - 1) - 3(y - 1) + 2(z - 1) = 0 \implies -x - 3y + 2z = -2. \]

7. We have
\[ \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}. \]

We compute
\[ \frac{\partial w}{\partial x} = \frac{2x}{x^2 - y^2 + z^2} - \frac{2(2s + t)}{(2s + t)^2 - (2s - t)^2 + 4st} = \frac{2(2s + t)}{12st} = \frac{2s + t}{6st}. \]

Similarly
\[ \frac{\partial w}{\partial y} = \frac{-2s - t}{6st} \]
\[ \frac{\partial w}{\partial z} = \frac{4\sqrt{st}}{12st} = \frac{1}{3\sqrt{st}}. \]

Next,
\[ \frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = 1, \quad \frac{\partial z}{\partial s} = 2 \cdot \frac{1}{2\sqrt{s}} \cdot \sqrt{t} = \sqrt{\frac{t}{s}}. \]

Therefore
\[ \frac{\partial w}{\partial s} = \frac{(2s + t)}{6st} \cdot 2 + \frac{(2s - t)}{6st} \cdot 2 + \frac{1}{3\sqrt{st}} \cdot \sqrt{\frac{t}{s}} = \frac{4t}{6st} + \frac{1}{3s} = \frac{1}{s}. \]

10. The derivative of the function \( f \) is given by
\[ Df = \begin{bmatrix} e^{x+y}(\sin(3x + 4y) + 3 \cos(3x + 4y)) & e^{x+y}(\cos(3x + 4y) - 3 \sin(3x + 4y)) \\ e^{x+y}(\sin(3x + 4y) + 4 \cos(3x + 4y)) & e^{x+y}(\cos(3x + 4y) - 4 \sin(3x + 4y)) \end{bmatrix}. \]

The determinant of this matrix is
\[ \det Df = e^{2x+2y}(\sin(3x + 4y) + 3 \cos(3x + 4y)) \cdot (\cos(3x + 4y) - 4 \sin(3x + 4y)) \]
\[ -e^{2x+2y}(\cos(3x + 4y) - 3 \sin(3x + 4y)) \cdot (\sin(3x + 4y) + 4 \cos(3x + 4y)). \]
This can be simplified by explicit calculation. We find
\[
\det Df = -e^{2x^2+2y}.
\]
Since the derivative cannot be zero, \( f \) is locally invertible.

14. The two constraints are
\[
g(x, y, z) = 1 - x^2 - (y - 1)^2 - z, \ h(x, y, z) = x^2 + y^2 - z.
\]
The function to be optimized is
\[
f(x, y, z) = x^2 + y^2 + (z - 1)^2.
\]
We set up the Lagrange multipliers. We have
\[
\nabla g = (-2x, -2(y - 1), -1), \ \nabla h = (2x, 2y, -1).
\]
These vectors could be linearly dependent. In this case, they are proportional, and the constant of proportionality is 1 by looking at the last component. Thus, we must have
\[
(-2x, -2(y - 1), -1) = (2x, 2y, -1).
\]
This means
\[
x = 0, y = 1/2,
\]
but no solution exists because the first constraint gives \( z = 3/4 \) while the second gives \( z = 1/4 \).
Therefore, we must have
\[
\nabla f = \lambda \nabla g + \mu \nabla h.
\]
This means
\[
(2x, 2y, 2(z - 1)) = \lambda(-2x, -2(y - 1), -1) + \mu(2x, 2y, -1).
\]
Thus
\[
2x = -2x\lambda + 2x\mu \implies x = 0 \text{ or } \mu - \lambda = 1.
\]
When \( x = 0 \), we have \( z = 1 - y^2 = 1 - (y - 1)^2 \) from the constraints, hence \( y = 1/2 \), and \( z = 3/4 \).
When \( \mu - \lambda = 1 \), we examine the remaining equations from Lagrange multipliers
\[
2y = -2\lambda(y - 1) + 2y\mu = 2y(\mu - \lambda) + 2\lambda = 2y + 2\lambda \implies \lambda = 0 \implies \mu = 1.
\]
Also,
\[
2(z - 1) = -\lambda - \mu \implies 2(z - 1) = -1 \implies z = 1/2.
\]
The constraints give \( x^2 + (y - 1)^2 = 1/2 \) and \( x^2 + y^2 = 1/2 \). This gives \( y^2 = (y - 1)^2 \implies y = 1/2 \) and thus \( x = \pm 1/2 \). The critical points are
\[
(\pm 1/2, 1/2, 1/2), \ (0, 1/2, 3/4).
\]

15.

(i) Clearly, \( \gamma \) is of class \( C^1 \). Next, \( \gamma \) is a \( 1 - 1 \) correspondence. Indeed, we can find the inverse of \( \gamma \) by solving
\[
x = u^5, \ y = uv, \ z = v^5 \implies u = x^{1/5}, \ v = z^{1/5}
\]
so
\[
\gamma^{-1}(x, y, z) = (x^{1/5}, z^{1/5}).
\]
Finally, we show \( D\gamma \) is injective. We have
\[
D\gamma = \begin{bmatrix}
5u^4 & 0 \\
u & u \\
0 & 5u^4
\end{bmatrix}.
\]
Injectivity simply means that the columns are linearly independent. If the columns are not linearly independent, then they must be multiplies of each other. This would imply \( 5u^4 = 0 \implies u = 0 \) and
\[ 5v^4 = 0 \implies v = 0. \] But the values \( u = v = 0 \) are not in the domain, hence our assumption was false, so \( D\gamma \) is injective. Thus \( \gamma \) is a parametrization.

(ii) The point \( (x, y, z) = (1, 2, 32) \) has \( u = 1, v = 2 \). The derivative equals

\[
D\gamma(1, 2) = \begin{bmatrix} 5 & 0 \\ 2 & 1 \\ 0 & 80 \end{bmatrix}.
\]

The tangent plane is spanned by the columns of this matrix.

(iii) For a critical point corresponding to a point with coordinates \( (x, y, z) = (u^5, uv, v^5) \) we must have that the tangent plane is contained in the null space of \( Df \), where

\[ f = x - 5y + z. \]

We have

\[ Df = \begin{bmatrix} 1 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 1 \end{bmatrix}. \]

The tangent plane is spanned by the columns of \( D\gamma \) so these columns must be in the null space of \( Df \). Thus when calculating \( Df \cdot D\gamma \) we should get zero:

\[
\begin{bmatrix} 1 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5u^4 & 0 \\ v & u \\ 0 & 5v^4 \end{bmatrix} = 0.
\]

This gives \( 5u^4 = 5v \) and \( 5u = 5v^4 \). Solving, we find \( u = v = 1 \) (since \( u, v > 0 \)). Thus \( (x, y, z) = (1, 1, 1) \) is the only critical point.