Math 31BH - Winter 2011 - Midterm I

Name: __________________________________________

Student ID: ______________________________________

Instructions:

Please print your name and student ID.

During the test, you may not use books or notes.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 5 questions which are worth 50 points, and a bonus question. You have 50 minutes to complete the test.

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Problem 1. [8 points]

For each of the following limits, either find its value or explain why it does not exist:

(i) [4]

\[
\lim_{{x,y \to 0}} \frac{{(x^2 + y^2)^2}}{{2x^2 + 3y^2}}
\]

(ii) [4]

\[
\lim_{{x,y \to 0}} \frac{{x^3y}}{{x^4 + y^4}}
\]
Problem 2. [16 points]

Consider the function

\[ f(x, y) = x^2 \sin(2y - 2x) \]

and the point \( P(1, \frac{\pi}{2} + 1) \).

(i) [4] Find the gradient of \( f \) at the point \( P \).

(ii) [4] Calculate the directional derivative of \( f \) at \( P \) in the direction \( \vec{u} = \frac{3i + 4j}{5} \).
(iii) [4] What is the unit direction of steepest decrease for the function $f$? What is the rate of decrease?

(iv) [3] Find a formula, valid for $(x, y)$ near $P$, which expresses the change $\Delta f$ in $f$ induced by a small change $\Delta x$ in $x$ and a small change $\Delta y$ in $y$.

(v) [4] Find the equation to the tangent plane to the graph of $f$ at the point $(P, f(P))$. 
Problem 3. [11 points]

Consider the function

\[ f(x, y) = \begin{cases} 
|y|, & \text{if } |x| \leq |y| \\
|x|, & \text{if } |y| < |x| 
\end{cases} \]

(i) [4] Draw the cross section of the graph of \( f \) corresponding to \( x = 1 \).

(ii) [4] Draw the contour diagram of \( f \) showing at least the curves for levels 1, 2 and 3.

(iii) [3] Draw (or, if you can’t draw nicely, at least describe) the graph of \( f \).
Problem 4. [9 points]

(i) [4] Consider the function

\[ f : \mathbb{R}^3 \to \mathbb{R}^2, \quad f(x, y, z) = (x + y^2, y + z^2). \]

Write down the total derivative of \( f \) at \((1, 1, 1)\) in matrix form.

(ii) [5] For Winter 2014: we did not cover the material relevant to this question. Find the tangent plane to the surface \( xy^3 + yz^2 + x^2 y = 3 \) at the point \((1, 1, 1)\). Estimate the value of \( z \) when \( x = 1.01 \) and \( y = 1.01 \).
Problem 5. [6 points]

(i) [4] Using greek letters, rigorously prove that the sequence

\[ x_n = \frac{2n^2 - 1}{n^2 + 1} \]

converges and find its limit.

(ii) [2] Complete the following phrase:

Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a function. We say that \( \lim_{x \to a} f(x) = L \) if . . .
Extra Credit. [5 points]

Show that if $K \subset \mathbb{R}^n$ is sequentially compact and $F \subset K$ is closed, then $F$ is sequentially compact as well.